

DOCUMENT RESUME

ED 095 881

IR 001 063

AUTHOR Kingsley, Edward H.; Stelzer, John
TITLE A Theoretical Basis for Individualized Instruction.
Final Report, March 1973-November 1973.
INSTITUTION Air Force Human Resources Lab., Brooks AFB, Texas.
REPORT NO AFHRL-TR-74-10; HumRRO-TR-74-9
PUB DATE Jul 74
NOTE 118p.
EDRS PRICE MF-\$0.75 HC-\$5.40 PLUS POSTAGE
DESCRIPTORS *Individualized Instruction; *Instructional Design;
Instructional Innovation; Mathematical Logic;
*Mathematical Models; Set Theory; Symbolic Language;
Training

ABSTRACT

A theoretical basis was formulated for a model of individualized instruction. The theory is semi-axiomatic in nature so that the definitions and assumptions used are stated explicitly. Set theory and symbolic logic are the conceptual tools used. The model includes theories of subject-matter structure and student state description. These are related by an overall instructional model. A main result shows how subject-matter structure constrains student state transitions through a subject matter. An application of the subject-matter theory is made to an existing Air Force course. A number of open problems are given whose further investigation would help make the model a more practical instructional tool. (Author)

A1

AFHRL-TR-74-10

AIR FORCE



**HUMAN
RESOURCES**

**A THEORETICAL BASIS FOR
INDIVIDUALIZED INSTRUCTION**

By

Edward H. Kingsley

John Stelzer

Human Resources Research Organization (HumRRO)
Alexandria, Virginia 22314

**TECHNICAL TRAINING DIVISION
Lowry Air Force Base, Colorado 80230**

July 1974

Final Report for Period March 1973 - November 1973

Approved for public release; distribution unlimited.

LABORATORY

**AIR FORCE SYSTEMS COMMAND
BROOKS AIR FORCE BASE, TEXAS 78235**

FD 095881

IR 001063

NOTICE

When US Government drawings, specifications, or other data are used for any purpose other than a definitely related Government procurement operation, the Government thereby incurs no responsibility nor any obligation whatsoever, and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise, as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

This final report was submitted by Human Resources Research Organization, 300 North Washington Street, Alexandria, Virginia 22314, under contract F41609-73-C-0020, project 1121, with Technical Training Division, Air Force Human Resources Laboratory (AFSC), Lowry Air Force Base, Colorado 80230. Captain William P. Mockovak was the contract monitor.

This report has been reviewed and cleared for open publication and/or public release by the appropriate Office of Information (OI) in accordance with AFR 190-17 and DoDD 5230.9. There is no objection to unlimited distribution of this report to the public at large, or by DDC to the National Technical Information Service (NTIS).

This technical report has been reviewed and is approved.

MARTY R. ROCKWAY, Technical Director
Technical Training Division

Approved for publication.

HAROLD E. FISCHER, Colonel, USAF
Commander

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFHRL-TR-74-10	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A THEORETICAL BASIS FOR INDIVIDUALIZED INSTRUCTION		5. TYPE OF REPORT & PERIOD COVERED Final March 1973 - November 1973
		6. PERFORMING ORG. REPORT NUMBER HumRRO-TR-74-7
7. AUTHOR(s) Edward H. Kingsley John Stelzer		8. CONTRACT OR GRANT NUMBER(s) F41609-73-C-0020
9. PERFORMING ORGANIZATION NAME AND ADDRESS Human Resources Research Organization (HumRRO) 300 North Washington Street Alexandria, Virginia 22314		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62703F 11210118
11. CONTROLLING OFFICE NAME AND ADDRESS Hq Air Force Human Resources Laboratory (AFSC) Brooks Air Force Base, Texas 78235		12. REPORT DATE July 1974
		13. NUMBER OF PAGES 122
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Technical Training Division Air Force Human Resources Laboratory Lowry Air Force Base, Colorado 80230		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
axiomatic methods education instructional models instructional theory	mathematical models military training set theory subject matter content	subject matter structure symbolic logic task structure
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This research was designed to formulate a theoretical basis for a model of individualized instruction. The theory is semi-axiomatic in nature so that the definitions and assumptions used are stated explicitly. Set theory and symbolic logic are the conceptual tools used. The model includes theories of subject-matter structure and student state description. These are related by an overall instructional model. A main result shows how subject-matter structure constrains student state transitions through a subject matter. An application of the subject-matter theory is made to an existing Air Force course. A number of open problems are given whose further investigation would help make the model a more practical instructional tool.		

SUMMARY

PROBLEM AND OBJECTIVE

An initial theoretical basis for a model of individualized instruction was developed and is presented in this report. Such a theory provides the general context required if educational technology is to proceed in a meaningful way.

According to Pask (1969), any complete theory of individualized instruction has to include, as components, representations of: (a) the subject-matter to be taught, (b) the educational goal, (c) the states of students, and (d) the teaching system. The theory discussed in this report gives an axiomatic formulation of the first three of these components and their interrelation.

APPROACH

The theoretical model presented in this report is axiomatic in nature. Primary concepts, axioms, definitions, and theorems of the theory are clearly stated, delineating the assumptions and starting points of the theory itself from the desired results.

Axiomatic treatments are well suited for characterizing empirical phenomena. Instruction can be viewed as a body of empirical phenomena, in the sense that the instruction process has developed over time in a trial-and-error, empirical manner. This being the case, the axiomatic method appeared to be well suited for the precise characterization of these phenomena.

RESULTS

A theory of subject-matter structure is presented first and includes both content and tasks as components. Subject-matter content is treated in terms of subject-matter constituents and relations between them. A task structure parallels the content structure and is related to the content structure by coordinating relations. The concept of dependency is defined and used to define the notion of precedence in an instructional sense.

A model is given to represent the state of a student at any point in time. A definition of the goal of instruction is formulated. The student state theory and the subject-matter structure theory are related by an overall theory of instruction. A major result shows how the student states are constrained by the subject-matter structure as the

student progresses from the initial state, through intermediate states to some terminal state. An algorithm is given for determining the possible paths through a subject-matter structure. Finally, the theory is applied to a concrete subject matter, a portion of the Air Force Apprentice Still Photographic Camera Specialist Course.

CONCLUSIONS

The concepts developed in this report provide an initial theoretical basis for a model of individualized instruction. Representations have to be developed of the teaching system and teaching strategies, Pask's last component, to complete the theory.

Even without this representation, the present form of the theory has a number of instructional ramifications that are obviously advantageous and others that are somewhat conjectural. Both of these types of instructional implications should be investigated further. In general, first a determination should be made of the difficulties encountered when the theory is applied to subject matters associated with complete courses. Second, a determination should be made of the effects of the theory on the instructional/learning process. Some of the problems that need to be explored are capable of being solved by theoretical means; others can be solved only by experiment.

PREFACE

This report describes the results of a study by the Human Resources Research Organization. The principal objectives of the study were to (a) formulate a theoretical basis for a model of individualized instruction, and (b) apply some of the theoretical concepts developed to a concrete example. This effort was successful in accomplishing these objectives; however, it should be emphasized that the model proposed is based on a theoretical approach which has not been validated within an Air Force technical training environment. Accordingly, both the theoretical approach and the instructional model need further development and empirical validation before they can be considered practical aids in the design and development of Air Force technical courses.

The study was conducted during March 1973-October 1973, by HumRRO Division No. 1 (Systems Operations), Alexandria, Virginia, under Air Force Contract F41609-73-C-0020. Dr. J. Daniel Lyons is Director of the Division. Dr. Robert J. Seidel is the Program Director, Instructional Technology Group. Mr. Edward H. Kingsley was the Principal Investigator and Dr. John Stelzer the Senior Staff Scientist. The report has been designated as HumRRO Technical Report 74-7.

CPT William P. Mockovak, USAF, Technical Training Division, Air Force Human Resources Laboratory (AFSC), was the Technical Monitor of the project and served as the principal point of contact during the period of the project. Dr. Gerry Deignan was the task scientist, and Dr. Marty Rockway the project scientist.

CONTENTS

	Page
INTRODUCTION	9
Chapter	
1 A THEORY OF SUBJECT MATTER STRUCTURE	13
The Structure of Subject Matter	13
General Cognitive Net (GCN) Description	17
Dependency	38
Tasks in Context Structure	42
2 A REPRESENTATION OF STUDENT STATES AND THE GOAL OF INSTRUCTION	48
Student State Descriptions	48
The Goal of Instruction	53
3 AN INSTRUCTIONAL MODEL	56
4 AN APPLICATION	66
TICS Development	97
Student State Descriptions	98
5 RELATION TO SOME OTHER THEORIES OF INSTRUCTION	100
The Merrill Theory	100
Relation of Subject-Matter Structure Theory to Merrill's Theory	101
6 RAMIFICATIONS	105
7 OPEN PROBLEMS	108
Application to a Full Subject Matter	108
Theoretical Problems	109
Instructional Implications	110
Experimental Aspects	111
References	113
Appendices	
A The Axiomatic Method	115
B Proofs of Theorems	118

LIST OF ILLUSTRATIONS

1 Factors contributing to subject matter structure.	16
2 Graphic representation of a simple GCN relation.	23
3 Graphic representation of a reflexive relation.	24
4 A chain of GCN relations.	24
5 Graphic representation of a single GCN constituent that depends on two other GCN constituents.	24

Illustrations (Cont.)

	Page
6 Non-admissible collection of GCN relations.	25
7 Example illustrating Theorem 4(i).	26
8 Example illustrating Theorem 4(ii).	26
9 Example illustrating Theorem 4(iii).	26
10 Example illustrating Theorem 4(iv).	27
11 GCN for explication of concept of a unicorn.	28
12 Graphic representation of the matrix in Table 2.	30
13 Graphic representation of the matrix in Table 3.	31
14 Graphic representation of the relation R	32
15 Alternative representation of the graph of the relation R	33
16 Graphic representation of the adjacency matrix of the relation S	35
17 Graphic representation of adjacency matrix $A(D) = [A(R) + A(S)] \#$	39
18 Illustration of a possible relation F_c for a set of three tasks.	45
19 GCN where c_1 depends on c_2	62
20 First step of the procedure.	62
21 Iteration 1 of second step of procedure.	63
22 Iteration 2 of second step of procedure: the possible paths through the GCN of Figure 19.	63
23 GCN where both c_2 and c_3 depend on c_1	64
24 The possible paths through the GCN of Figure 23.	65
25 GCN where c_3 depends on both c_1 and c_2	65
26 The possible paths through the GCN of Figure 25.	65
27 Graphic representation of the "used in the formulation of" relation.	74
28 Graphic representation of the "used to establish" relation.	75
29 Adjacency matrix for the "used in the formulation of" relation.	77
30 Adjacency matrix for the "used to establish" relation.	78
31 Adjacency matrix for the "used in the formulation of" relation R	79
32 Adjacency matrix for the "used to establish" relation S	80
33 Adjacency for the dependency relation in the photo course GCN, $A(D)$	81
34 $A^2(D)$	82
35 $A^3(D)$	83
36 $A^4(D)$	84
37 $A^5(D)$	85
38 $A^6(D)$	86
39 $A^7(D)$	87
40 $A^8(D)$	88
41 $A^9(D)$	89
42 $A^{10}(D)$	90
43 $A^{11}(D)$	91
44 $A^{12}(D)$	92

Illustrations (Cont.)	Page
45 $A^{13}(D)$	93
46 $A^{14}(D)$	94
47 $A^{15}(D)$	95
48 Computer dependency matrix $\left[\begin{array}{c} \Sigma \\ i = 1, 15 \end{array} A^i(D) \right]$	96
49 Ten categories of learned behavior (from Merrill, 18).	101

LIST OF TABLES

1 Terminological Correspondence	18
2 Basic Principles Used in Formulating Primary Notions	29
3 Primary Notions and Secondary Notions Used in Formulating Secondary Notions	30
4 Primary Notions and Secondary Notions Used to Formulate Established Principles	31
5 Adjacency Matrix $\underline{A}(\underline{R})$ of the Relation $\underline{R}$	32
6 Adjacency Matrix $\underline{A}(\underline{S})$ of the Relation $\underline{S}$	34
7 Adjacency Matrix $\underline{A}(\underline{D}) = [\underline{A}(\underline{R}) + \underline{A}(\underline{S})]$ #, for Dependency Relation $\underline{D}$	38
8 Dependency Matrix, $\underline{A}^2(\underline{D})$ #	39
9 Dependency Matrix $\underline{A}^3(\underline{D})$ #	40
10 Dependency Matrix $\underline{A}^*(\underline{D}) = [\underline{A}(\underline{D}) + \underline{A}^2(\underline{D}) + \underline{A}^3(\underline{D})]$ #	40
11 The Matrix $\hat{\underline{A}}^2(\underline{D})$ # Giving the Number of Paths of Exactly Length 2	41
12 The Matrix $\hat{\underline{A}}^3(\underline{D})$ Giving the Number of Paths of Exactly Length 3	41
13 Number of Paths of Length 3 as Taken From Figure 17	41
14 Tabular Representation of the "Used in the Formulation of" Relation	72
15 Tabular Representation of the "Used to Establish" Relation	73
16 Application of Subject-Matter Constituents to Merrill's Taxonomy	102
17 Application of Subject-Matter Constituents and Merrill's Four Levels of Behavior	103

INTRODUCTION

A great deal of empirical work has been done in the area of individualized instruction, particularly in the area of Computer Assisted Instruction (CAI). However, this work has been limited because little serious effort has been directed to the development of a comprehensive theory underlying individualized instruction. Educational technology must proceed within the context of a theory of educational technology if it is to advance in a meaningful way. Formulation of such a theory is an essential preliminary to any successful pursuit of technology.

Pask (1969) has given the ingredients of an individualized instructional environment in the form of five components that should be present in any complete theory of individualized instruction. His five components are:

- (1) A representation of the subject matter to be taught.
- (2) A representation of the educational goal.
- (3) A representation of the initial or starting state of the student entering the system.
- (4) A representation of the current student state, or the state of the student at any time.
- (5) A representation of the teaching system including teaching strategies.

The objective of this study is to develop an initial theoretical basis for a model of individualized instruction by concentrating on the first four of Pask's components and their interrelations. The approach taken in the construction of the theory is summarized by the term "explication" as used by Carnap; briefly, the task of explication is the search for a transformation of a given more or less inexact concept into an exact one—the replacement of an inexact concept by an exact concept. The inexact concept can be expressed by everyday language or by a previous stage in the development of scientific language. On the other hand, the exact concept must be given by explicit rules for its use. The end product of an explication is a formal theory.

The many inexact concepts used in education were probably sufficient for the educational technology present before the modern digital computer and its auxiliary devices. However, with modern technology, the continued presence of inexact concepts in

education can no longer be tolerated. Work on explicating inexact concepts has to be started before substantive progress can be made in building the rigorous theory of individualized instruction demanded by educational technology. This report presents a formulation of such a theory.

The theory to be presented is semi-axiomatic in nature. Two things are required to axiomatize a theory: a body of relevant scientific knowledge and an understanding of the technique of theorizing. A short introduction to the axiomatic method is given in Appendix A. Very briefly, to axiomatize a body of knowledge, a few terms are selected and considered to be undefined. Other terms of the body of knowledge are then defined by using those undefined terms and previously defined terms. In a similar manner, a few statements asserted in the body of knowledge are taken as true without trying to establish their validity and are called axioms. All other statements in the body of knowledge are then accepted as being true only if their validity can be established by legitimate arguments using nothing but axioms, definitions, and previously established statements.

In the first three chapters of this report, the first four components in Pask's list are modeled and integrated. Definitions, axioms, and theorems are presented formally and discussed informally with the help of examples; proofs of theorems are given separately in Appendix B. Chapter 4 contains a comprehensive detailed example that serves as a vehicle to illustrate the theoretical concepts, results, and procedures, and Chapter 5 provides a short discussion of the relation of the presented theory to other theories of instruction. Chapters 6 and 7 conclude the report with remarks on the instructional implications of the theory and open problems.

The first component in Pask's list is considered in Chapter 1, and an axiomatic theory of subject matter structure presented. The subject-matter content is analyzed in terms of constituents and their relations; it includes both content and tasks as components. Paralleling the content structure is the task structure and the two structures interact by coordinating relations. The notion of dependency is introduced and investigated. Dependency leads to precedence in an instructional sense and this notion is also discussed. Formal procedures are developed that can be used to formulate the complete dependency relationship for particular subject matters. Chapter 1 is the longest chapter in the report, reflecting the integrative influence a theory of subject-matter structure has on all aspects of a theory of instruction.

Pask's second, third, and fourth components are modeled in Chapter 2. Using symbolic logic and set theory, representations are given for describing the state of a

student at any time and for describing the goal of instruction. The theory presented in this chapter is an extension of the axiomatic subject matter theory given in Chapter 1.

In Chapter 3, the subject-matter structure theory and the student state description theory are related in terms of an overall instructional model. A main result of the chapter is to show how the subject-matter structure constrains the student state transitions as the student passes from the initial state, through intermediate states to some terminal state. A procedure is given for determining the possible paths through a subject-matter structure.

A concrete example is given in Chapter 4 to illustrate the theoretical concepts presented in the first three chapters. An application is made of the theoretical model to the AF Apprentice Still Photographic Camera Specialist Course.

Pask's last ingredient in his list is not discussed in an axiomatic manner as was done for his first four ingredients. An axiomatization could be given of a teaching system, but present knowledge does not allow the defense of choosing one teaching system over another. Thus, instead of continuing the axiomatic theory, Chapter 5 deals with the relation between the theory in the report and some current theories of instruction.

Instructional ramifications of the theory are discussed in Chapter 6 and are related to the open problems discussed in Chapter 7. Some of the ramifications are conjectural and require further investigation for clarification. Ramifications of this type generate the "open problems" discussed in Chapter 7. One obvious ramification is that the only environment in which the entire theory is realizable is a CAI environment. On the other hand, portions of the theory can be used on subject-matters presented conventionally. The subject-matter structure theory is an example of one of these portions.

Chapter 1

A THEORY OF SUBJECT MATTER STRUCTURE

A representation of the subject matter to be taught is one of the five components listed by Pask (1) as necessary for any comprehensive model of instruction. To represent subject matter requires the subject matter be organized or "structured." A main objective of this chapter is to formulate a definition of the "structure" of a subject matter. The definition proposed is an implicit definition and is contained in a general theory developed to provide a framework for structuring subject matter. A means of representing and describing such structures will be given also. For a more detailed treatment of the material in this chapter, see Kingsley and Stelzer (2).

THE STRUCTURE OF SUBJECT MATTER

The necessity for a theory of subject matter structure becomes apparent when the problem of using computers as instructional agents is studied. Need for such a theory is particularly important when Computer Administered Instruction (CAI) Systems are involved. A theory of subject matter structure provides a foundation upon which all the components of a theory of instruction can be based, permits study of the order in which things should be taught, and serves as a map for the instructional agent.

The theory of subject matter structure presented in this chapter is inspired by the example of axiomatized mathematics. In short, the axiomatic method is viewed as an organizer of bodies of knowledge. Implicitly, to axiomatize a theory related to a subject matter is to structure it. This does not imply every subject matter is reducible to a mathematical, let alone an axiomatic mathematical, equivalent. We simply believe the axiomatic method itself provides an example of an approach that can be more generally applied to nonmathematical topics.

Historically, attempts to characterize subject matter structures seem to fall into one of two classes: (a) Theories such as Gagné's (3, 4, and 5) have focused on task structures. Gagné's formula, "What must one be able to do before . . .?" leads to structures that

focus on the tasks a student must learn how to perform. (b) Pask (1), on the other hand, has focused on the content of subject matter. Pask structures subject matter along the lines of the formula: "What must a person know before he can learn . . .?"

It seems clear that an adequate theory of subject matter structure must include both task-oriented and content-oriented components. Consider the problem of specifying tasks. In order to specify tasks, the behavioral objectives of the instruction must be known. These objectives include terminal behavior, the specification of which has been called "the beginning of any design for programmed instruction" (Tennyson and Boutwell, 6). Tuckman and Edwards (7) refer to the process of specifying behavioral objectives in terms of tasks or task analysis. In general terms, the specification of terminal behavior, behavioral objectives, etc., can only be done from the point of view of the *context* of the instruction. That is, in order to specify the behavioral objectives and tasks, information concerning certain aspects of the instructional environment must be known. Relevant mediating considerations include the following: the target population, abilities that can be assumed on the part of the target population, the general purpose of the instruction, the general interests of the target population, and so forth. This aspect of instruction is referred to as the *context* of the instruction.

In addition to the context, a general specification of the subject matter, *General Subject Matter*, must be given. In this sense, the statement of the General Subject Matter begins to delineate the content of a discipline. A discipline can be denoted by general terms, such as the discipline called mathematical statistics. In this case, the content of the discipline is large and varied. If a more specific discipline is being considered, such as the discipline called multivariate analysis, the content is smaller and less varied. Although a General Subject Matter has some structure, the structure is difficult to define. As will be seen, it begins to take on more definite structure after other factors (e.g., the context) are considered.

Applying the General Subject Matter to a specific context results in a specification of the content of the instruction: the *Context Conditioned Subject Matter*. As an example, suppose the General Subject Matter is taken to be the content of an undergraduate first course in statistics. Then the content of such a course might differ when given under the contexts associated with groups of students whose primary interests are in the disciplines of mathematics, psychology, economics, business, engineering, biology, or medicine. Each of these disciplines modifies the content of the General Subject Matter "statistics" to make it relevant to the discipline and to the target population. In many instances, the differences in content result from using the terminology peculiar to a

discipline and to the target population. In other instances, the differences in content result from using the terminology particular to a discipline in presenting examples and exercises. Other less trivial differences arise by including special techniques useful in particular disciplines. Specifying contexts thus partially defines subsets of the General Subject Matter. In a general way, the boundaries and content of these subsets begin to be identified.

So far, we have argued that the content of a General Subject Matter must be considered in contexts and never in isolation. A Context Conditioned Subject Matter begins to have structure if, and only if, its constituents are listed and the ways in which they are interrelated are displayed. Structure always involves one or more relations. A mere listing of constituents as a class has no structure. Thus, to say a Context Conditioned Subject Matter has structure is, first, to identify and to list its constituents and, second, to specify the relations, and their type, between constituents.

Because the properties of constituents from a Context Conditioned Subject Matter are related to each other by one or more relations, it is useful to think in terms of graph theory and to depict the structure of a subject matter by graphs. The nodes of the graph correspond to constituents or to their properties, and the arcs of the graph represent relations among the nodes. Because several relations among the properties are possible, such a diagram is called a *net* instead of a graph (as noted in Hohn *et al.*, 8). Thus a *General Cognitive Net* (GCN)¹ is defined as the graphic representation of the constituents and their relations from a Context Conditioned Subject Matter.

Given the General Subject Matter and Context of the instruction, it is also possible to specify the tasks to be performed as a result of the instruction: the *Context Conditioned Behavioral Objectives*. These objectives include both the enabling and the terminal objectives of the instruction. The behavioral objectives are stated by explicitly defining the tasks that the instructor expects the students to perform and the criteria for each task.

The set of tasks defined by the behavioral objectives are not totally unrelated to one another and, thus, have structure. A student may have to demonstrate ability to perform certain previous tasks before he is permitted to undertake other tasks. For example, consider a course in statistics. In such a course, a student might be expected to perform to some criteria level of competence tasks concerning the interpretation of the area under portions of a univariate density function, before undertaking tasks on interpreting the

¹ Dr. Robert Seidel of HumRRO contributed significantly to the original versions of the content/task (GCN/TICS) analysis of subject matter structure.

volume under a portion of a bivariate density function. In such cases, the respective tasks are related by a precedence relationship. A further relation can be defined between the subtasks of a task by including considerations of subtask complexity. Hence, the tasks are structured through relationships, and we call the task structure the *Task In Context Structure* (TICS).

This discussion is summarized in Figure 1. The six upper blocks in the diagram correspond to the six concepts just described. An arrow goes from one block to another block if, and only if, a change in the state of the first block has an immediate effect on the state of the second block. The diagram is related to Ashby's (9) diagram of immediate effects.

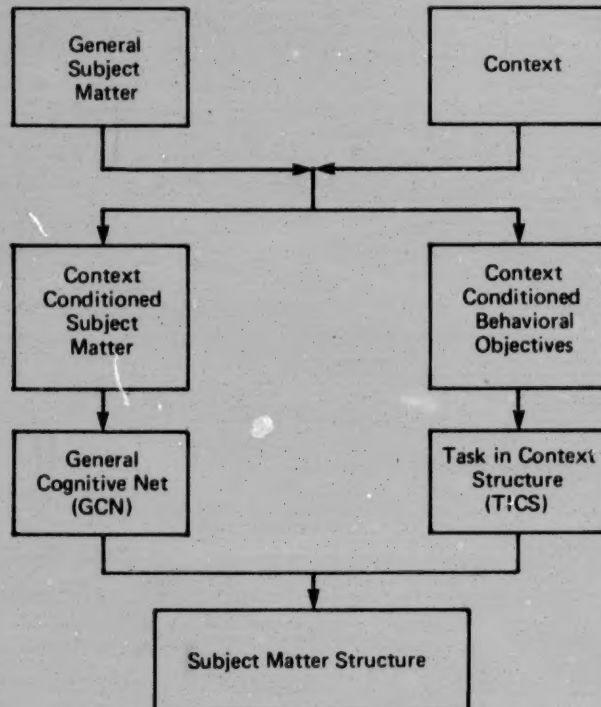


Figure 1. Factors contributing to subject matter structure.

We have made a distinction between properties of constituents (the GCNs) and learning these properties, on one hand, and tasks and problem solving (the TICS) on the other. In short, properties of constituents are defined independently of their utility in solving specific problems, while tasks refer to the groupings of properties of constituents into utilitarian (pragmatic) units to solve particular problems.

Using pictorial terms, visualize at least two planes in space: the net-like structure of properties of constituents (the GCNs) is drawn on one plane, and the net-like task structure (the TICS) on the other. Lines go from single elements of the GCNs to elements of the TICS. These lines depict the relation between the two structures, and their aggregate is termed the *Coordinating Relations*. We call the entire complex of nets and Coordinating Relations between the nets the *Subject Matter* structure. Both the cognitive structured understanding (obtained from the GCNs) and the applications (obtained from the TICS) are important for proper problem-solving behavior. Said in another way, together, they represent the necessary and sufficient conditions for solving problems pertinent to a given suitably restricted subject matter.

As an example, consider mathematical statistics as the subject matter. Further, suppose it has been modified to reflect a context and a set of behavioral objectives. Then, to have had instruction solely on how various statistical terms relate to one another without actually having solved specific empirical problems would result in one's "knowing" statistics (understanding the relations among the terms), but not in one's being able to do empirical tasks adequately. On the other hand, performing empirical tasks without also obtaining instruction on the cognitive structure would probably lead to failure to perform as soon as some verbal categories were changed to the unfamiliar. Both skills involve information processing. The processing of the task-specific requirements results in many connections relating verbal (cognitive) representations to other specific, physical acts.

In the general setting of a model for individualized instruction, Figure 1 and the above discussion are basically a prescriptive aspect of the model. Figure 1 summarizes guidelines to be followed in the development of subject matter structures. As such, it is not descriptive, because it does not describe in detail the actual subject matter structure. In the next section, we take up the problem of developing the descriptive portion of the model for structuring subject matter.

GENERAL COGNITIVE NET (GCN) DESCRIPTION

The theory we propose for structuring the GCN parallels closely the axiomatic method discussed in Appendix A. Thus, the GCN will contain elements analogous to primitive and defined concepts, axioms, and theorems in axiomatics. In order to discuss these elements, while at the same time distinguishing them from components in axiomatic

systems, specialized terminology is necessary. To this end, the categories of GCN elements will be referred to as Primary Notions, Secondary or Defined Notions, Basic Principles, and Established Principles. The correspondence between axiomatic and GCN terminology is displayed in Table 1.

Table 1. Terminological Correspondence

Axiomatics	Subject Matter
Primitive Concept	Primary Notion
Defined Concept	Secondary Notion
Axiom	Basic Principle
Theorem	Established Principle

An axiomatic theory of subject matter structure will be developed and the axioms, definitions, theorems, and so forth, will be introduced, providing a description of the theory. Since the subject matter structure theory is similar to an axiomatic theory, in a sense an axiomatic treatment of axiomatics is provided. The standard convention of first discussing the Primary Notions in the theory will not be followed. They will be introduced in the axioms; they will be apparent and will be discussed as appropriate.

Now consider the first assumption made concerning the GCN.

Axiom 1. The GCN contains a finite set of distinguished notions called the Primary Notions for the subject matter. These notions are basic in that they depend on no others for their development or understanding.

Axiom 1 asserts that once the context is identified in which a body of knowledge is to be taught, it is possible to isolate or identify some basis or starting point for instruction.

Notice that Axiom 1 does not prescribe how to identify or choose the Primary Notions. As noted in Appendix A, in axiomatic theories there is considerable latitude in selecting primitive concepts. Similarly, in selecting Primary Notions for instructional purposes, there is a great deal of freedom of choice. The notions chosen as primary vary depending upon the bias of the individual structuring the subject matter, the applications to be made of the knowledge to be learned, and so forth. Consider, for example, set theory as a potential instruction subject. One instructor might select the following Primary Notions:

- set
- set membership
- the empty set
- set complementation
- set intersection

while another instructor might select as a possible set of Primary Notions:

set
set membership
the universal set
set complementation
set union

Each of these two classes of notions can serve as the bases for instruction in set theory. Notions present in one class, but absent from the other, can be introduced by way of definition on the bases of the other notions. For example, set intersection in the first class can be defined on the basis of the second class by means of set union and set complementation. The point is, there is no compelling abstract reason to select one class of Primary Notions over the other.

Analogous to introducing definitions in axiomatic theories, the GCN also includes a set of Secondary Notions. Thus,

Axiom 2. The GCN contains a finite set of Secondary Notions that depend on Primary Notions or previously introduced Secondary Notions.

It is possible for the set of Secondary Notions to be empty. In this case the subject matter to be taught consists only of the Primary Notions and any principles that follow from them. Ordinarily, the set of Secondary Notions is not empty, since these elements are used to group notions in the subject matter into other meaningful notions.

A GCN contains both Primary and Secondary Notions. These elements are used to formulate basic statements concerning the concepts. These basic statements are regarded as self-evident truths, hence establish accepted principles. As such, they correspond to axioms in axiomatic theories. These elements of the GCN as shown in Table 1 will be referred to as Basic Principles of the particular subject matter, and their existence is ensured through the following axiom.

Axiom 3. The GCN contains a finite set of statements called the Basic Principles that are assumed to be true and are not dependent on any other established results.

As with the choice of Primary and Secondary Notions, there is a great deal of latitude in selecting Basic Principles for a particular subject matter. In general, the Basic Principles provide implicit definitions of the Primary Notions. Therefore, the choice of the Primary Notions dictates to some extent the Basic Principles that will be given. In particular, it is expected that all the Primary Notions will appear at least once in the set of Basic Principles. If this is not the case, then some Primary Notion is not being utilized in the primary role for which it is intended. In this case, the notion under consideration could be deleted with no apparent loss to the subject matter formulation.

Analogous to deriving theorems in axiomatic theories, the Primary and Secondary Notions, in conjunction with the Basic Principles, are used to establish results in the GCN, which, in turn, can be used to establish further results. Thus,

Axiom 4. The GCN contains a set of statements called the Established Principles that depend on the Primary Notions, Secondary Notions, Basic Principles, and preceding Established Principles.

As with the Primary and Secondary Notions, and the Basic Principles, the Established Principles must be chosen. The instructor is free to do so depending upon the purposes and goals of the instruction.

To summarize, Axioms 1 through 4 assert that any GCN comprises Primary and Secondary Notions, and Basic and Established Principles. Together these elements provide the basic components of the GCN. We call these components constituents and introduce this notion formally.

Definition 1. The constituents of the GCN consist of the Primary Notions, Secondary Notions, Basic Principles, and Established Principles.

The constituents in the GCN are assumed to be interrelated. That is, it is assumed that a collection of relations exists in the GCN that serve to establish connections between the GCN constituents. Consistent with the foregoing, these relations cannot be specified abstractly. The relations included in any particular GCN will be determined by the instructor depending upon the purposes and goals of the instruction. However, as will be seen, the relations will be assumed to satisfy certain structural properties.

To discuss relations in precise terms, it is convenient at this point to introduce some standard formal notation. Let x and y be constituents in a GCN, and let R be a relation defined on the GCN. Rxy will be taken to mean x is related to y by R . For example, if R is the relation "is the father of," then Rxy asserts that x is the father of y . In general, capital English letters will denote GCN relations and lower case English letters will denote GCN constituents. Subscripts will be used occasionally to distinguish relations and constituents.

As with the Primary Notions, the individual structuring the subject matter has a great deal of freedom in choosing what notions to introduce and the order in which to introduce them. In the context of the set theory example, one instructor may introduce as Secondary Notions:

ordered pair
cross product
relation
function

Another instructor, however, may forego the relation concept and introduce the notion of function based upon an understanding of ordered pairs and cross product. Thus, the class of Secondary Notions is reduced in this case to:

ordered pair
cross product
function

The difference in approach might be dictated by the context of the instruction with the first class being more appropriate for social scientists where relations are of importance. The second class of Secondary Notions may be more appropriate for mathematically oriented instruction emphasizing the theory of functions.

Notice also that the set of Primary Notions and the set of Secondary Notions are both finite. This assumption is made to ensure that the subject matter could indeed be mastered in a finite length of time. That is, it is not possible to receive instruction on an infinite number of distinct notions in a given subject matter.

Before continuing, a few comments are given concerning the basic tools used in developing the theory. The tools are symbolic logic and set theory. For reference, our notational conventions are those found in Suppes (10). The symbols, $\neg$ and $\&$, represent "negation" and "conjunction." Following standard practice, universal quantifiers are suppressed for ease of reading.

The axiom to be used to establish the existence of GCN ordering relations is

Axiom 5. There is a finite, non-empty class of binary relations, called GCN relations, defined on the set of GCN constituents.

Axiom 5 is completely neutral with respect to what relations are actually used to structure the GCN. This theoretical approach is a radical departure from other theories of subject matter structure. Some authors have attempted to develop prescriptions for structuring subject matter that are applicable to a wide range of subject matters (Gagné and Paradise, 11).¹ Most of these approaches are intended to provide unique subject matter structures. It is implicit in Axiom 5 that a unique structure for a particular subject matter is not forthcoming. The particular structure that will emerge for a given subject matter depends upon which Primary Notions are chosen, which Basic Principles are chosen, and the order in which Secondary Notions and Established Principles are introduced and established.

One additional point is that traditionally subject matter structures have been developed on the basis of one relationship. Gagné's structure theory is an example of a single

¹ Pask, G. "Structuring Subject Matter With Multiple Relations," a paper presented at a conference at HumRRO, Alexandria, Virginia, 1971.

relation approach. Recent work by Pask¹ is an exception to this rule. Pask has attempted to structure subject matter with at least two relations. While we believe that any theory for structuring subject matter must permit multirelational structuring, we do not believe the number of relations should be restricted to two, three, or any other fixed number. The number of relations required to structure a given subject matter will depend, at least, on the subject matter, the instructor, the learner population, the purpose of the instruction, and so forth.

For technical reasons, properties of GCN relations will be deduced from the properties of one very general GCN relation that includes GCN relations as particular instances. The general relation we use rests on the notion of one GCN constituent depending upon another. The dependency relation has been used informally in stating the first four axioms. The symbol D will be used to denote the dependency relation. Furthermore, Dxy will mean, "y depends on x" or "x is required for y." The formal definition of the dependency relation is given by:

Definition 2. Let x and y be two GCN constituents, then y depends on x if and only if there are n GCN constituents $x_1, x_2, \dots, x_n$ and $n+1$ atomic GCN relations $R_1, R_2, \dots, R_{n+1}$ such that $R_1xx_1 \ \& \ R_2x_1x_2 \ \& \ \dots \ \& \ R_{n+1}x_ny$.

Thus, for y to depend on x means that there exists a relational chain going from x to y . The final expression in Definition 2 asserts: x is related by R_1 to x_1 , and x_1 is related by R_2 to x_2 , and so on, so that finally x_n is related by R_{n+1} to y . Finally, note that the dependency relation is defined in terms of the previously postulated atomic GCN relations.

The dependency relation reduces to a GCN relation between x and y when only one GCN relation is present and there are no intervening GCN constituents between x and y .

A fundamental property of the dependency relation is given by:

Axiom 6. The dependency relation is asymmetric.

To recall, a binary relation is said to be asymmetric if whenever the relation holds between two constituents x and y then it is impossible for the relation to hold between y and x . For example, if F is the relation "father of," and if Fxy (x is the father of y), then it is false that Fyx . Symbolically expressed, for the dependency relation D , if Dxy then $\neg Dyx$.

¹ Ibid. Footnote, p. 10.

From Axiom 6 and Definition 2, the dependency relation can be shown to have the following two properties:

Theorem 1. The dependency relation is

- (i) irreflexive
- (ii) transitive.

In more detail, if D were reflexive then a GCN constituent x would depend upon itself. In symbols, we would have Dxx . Not being reflexive means that Dxx is false or $\neg Dxx$. The transitivity of the dependency relation means that if y depends on x and if z depends on y , then z depends on x . Symbolically written:

$$Dxy \ \& \ Dyz \text{ imply } Dxz.$$

The properties of the dependency relation given by Axiom 6 and Theorem 1 are the properties we intuitively expect. In more condensed form, the dependency relation is a strong ordering (Gleason, 12) of the GCN constituents.

The dependency relation is too general for structuring subject matter. More specific GCN relations are required for subject matter structuring. The properties given, by Axiom 6 and Theorem 1, of the dependency relation can be used to establish properties that restrict GCN relations. These restrictions on the GCN relations then impose a structure on the subject matter. To begin with, we can prove:

Theorem 2. Every GCN relation is

- (i) irreflexive
- (ii) asymmetric.

We now see that the GCN consists of constituents and relations defined on the set of constituents. This leads to a natural graphical representation of the GCN in which constituents are represented by nodes, and directed lines connecting nodes represent the relations. For example, Figure 2 depicts a simple GCN. The nodes of the GCN are x_1 and x_2 . One relation is defined in the GCN, and x_1 and x_2 are related through this relation. As shown in the figure, a directed line representing the GCN relations is required because of the asymmetry constraint imposed by Theorem 2.

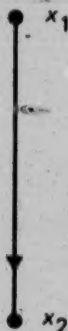


Figure 2. Graphic representation of a simple GCN relation.

If x is a GCN constituent and R a GCN relation, then the graph shown in Figure 3 is associated with a reflexive relation R . But, R is not reflexive, so that such relations are not present in a GCN.



Figure 3. Graphic representation of a reflexive relation.

To summarize, for y to depend on x means that there is a set of constituents $x_1, x_2, \dots, x_n$ and a chain of GCN relations $R_1, R_2, \dots, R_{n+1}$ leading from x to y . For example, note the situation shown in Figure 4 in which y depends on x .

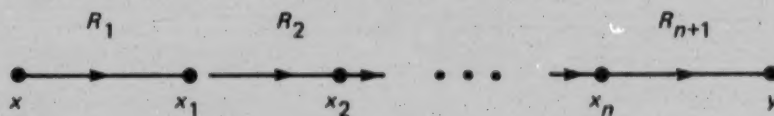


Figure 4. A chain of GCN relations.

Notice that x and y need not be related by a single GCN relation in order for dependency to hold. There only needs to be a *chain* of GCN relations from x to y . As an example of a dependency relation, consider a simple GCN with three constituents, two of which are Primary Notions, and the third, a Secondary Notion. Let the single GCN relation be, "is used in the formulation of." Assume that both of the Primary Notions x_1 and x_2 are used in the formulation of the Secondary Notion x_3 . Figure 5 shows the situation. The directed lines show that Rx_1x_3 and Rx_2x_3 both hold. From Axiom 6, it can be seen that x_3 depends on both x_1 and x_2 .

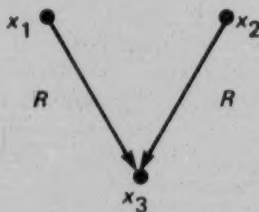


Figure 5. Graphic representation of a single GCN constituent that depends on two other GCN constituents.

The next theorem rules out GCN relations with a graph in the form shown in Figure 6.

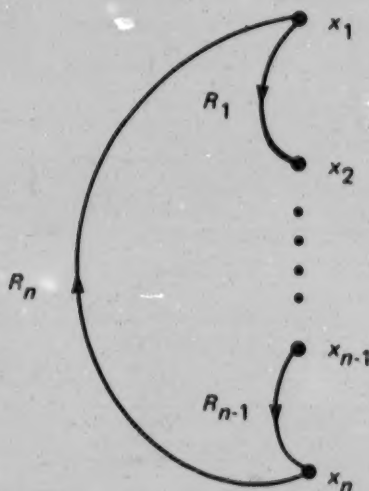


Figure 6. Non-admissible collection of GCN relations.

Theorem 3. If $x_1, x_2, \dots, x_n$ are GCN constituents and $R_1, R_2, \dots, R_n$ are GCN relations, then it is impossible to have a sequence of the form

$$R_1 x_1 x_2 \ \& \ R_2 x_2 x_3 \ \dots \ \& \ R_n x_n x_1.$$

Notice, in particular, that Theorem 3 rules out the possibility of Rxy and Syx holding for GCN constituents x, y and GCN relations R, S . This means that if x is related to y by one relation, then y cannot be related to x by another relation. As an example, let R and S be the following two relations;

R "is used in the definition of"

S "is used in the derivation of."

Then, if x and y are two GCN constituents, it is impossible for both of the statements, " x is used in the definition of y " and " y is used in the derivation of x ," to be simultaneously true.

In general, Theorems 2 and 3 rule out the possibility of loops in the GCN structure. In particular, as shown by Theorem 2, a single relation also cannot generate such a loop.

The implicit meaning of the terms Primary Notion, Secondary Notion, Basic Principle, and Established Principle as used in the GCN axioms can be given in terms of the GCN relations. This is done in the following theorem.

Theorem 4. (i) Let x be a Primary Notion in a GCN. Then $\neg Rxy$ for every constituent y and relation R in the GCN.

- (ii) Let y be a Secondary Notion in a GCN. Then Rxy holds for some constituent x and some relation R in the GCN, where x is a Primary Notion or a Secondary Notion.
- (iii) Let x be a Basic Principle in a GCN. Then $\neg Ryx$ for every constituent y which is a Basic Principle or Established Principle in the GCN, and every GCN relation R .
- (iv) Let y be an Established Principle in a GCN. Then Rxy holds for some constituent x and some relation R in the GCN.

The content of each of the parts of Theorem 4 can be depicted by way of graphs. Part (i) asserts that all relations point away from Primary Notion x as shown in Figure 7.

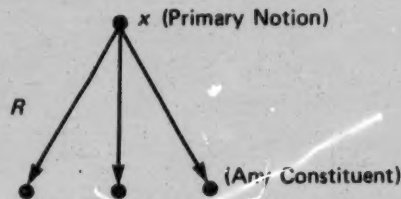


Figure 7. Example illustrating Theorem 4(i).

That is, in graph theoretic terms, a Primary Notion cannot have arcs leading into it. A Secondary Notion, on the other hand, must have at least one arc leading into it—Part (ii) of the theorem, as illustrated in Figure 8.

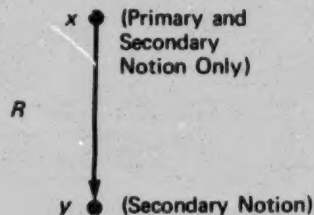


Figure 8. Example illustrating Theorem 4(ii).

By part (iii), only Primary and Secondary Notions can lead into a Basic Principle as shown in Figure 9.

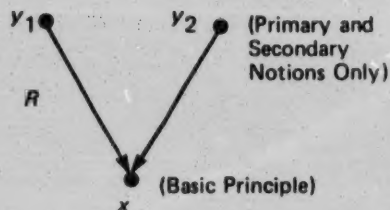


Figure 9. Example illustrating Theorem 4(iii).

Finally, Part (iv) asserts that an Established Principle must be dependent on some other GCN constituent as shown in Figure 10.

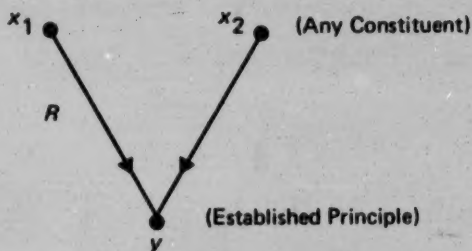


Figure 10. Example illustrating Theorem 4(iv).

Notice that, as expected, a Basic Principle qualifies as an Established Principle, but not conversely.

It must be stressed that Theorem 4 strongly constrains the structure imposed upon the GCN. Parts (i) and (iii) rule out certain relationships in the structure. Parts (ii) and (iv), however, assert that some relational properties *must* exist in the structure. For example, if some notion is to be included in the GCN as a Secondary Notion, then Theorem 4 asserts that there *must* be a GCN relation that relates this element to another. This might seem to be a severe restriction; however, it is reasonable since this is what is meant to introduce a notion as being secondary. It does compel the individual developing the GCN to provide the required relation.

At this point, two examples of the GCN theory will be developed. The first example is intended to provide a simple overview of the theory, and, thus, a trivial subject matter has been selected.

GCN Example 1

For the first example, suppose the instructional problem is to provide an explication of the meaning of the term "unicorn." The General Subject Matter is that concerning unicorns. Assume the target population consists of a group (maybe only one) of youngsters reading a fairy tale. The Context Conditioned Subject Matter consists of explicating the concept of unicorn with respect to the knowledge that the youngsters possess, and perhaps the particular fairy tale being read. It is assumed that the students know what a horse, a stag, a lion, and horns on animals are.

The instructor in this case may proceed as follows. The Primary Notions can be chosen as being the notions of animal, horse, stag, lion, and horn. From these, Secondary Notions can be introduced. First is that of an animal with a horse's head and body.

Second is that of an animal with a horse's head, horse's body, and stag's legs. Third is that of an animal with a horse's head, horse's body, stag's legs, and lion's tail. Finally, this notion can be extended to that of a creature with a single horn in addition. Therefore, the GCN is structured as shown in Figure 11.

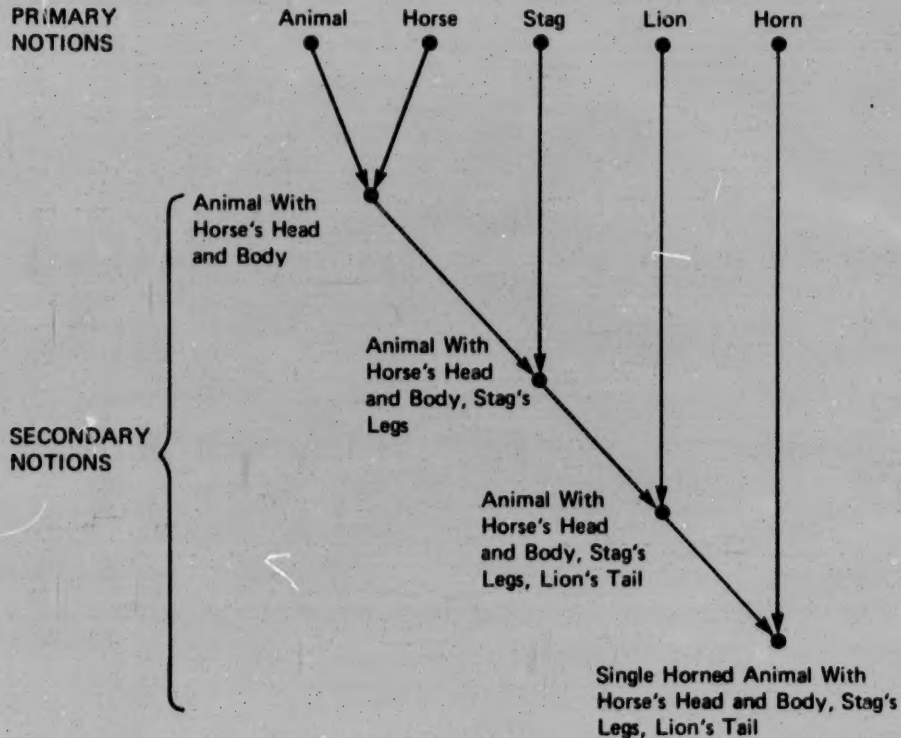


Figure 11. GCN for explication of concept of a unicorn.

The relation R in this example is simply the relation "used in the explanation of." For example, let x be the notion of an animal, and let y be the notion of an animal with a horse's head and body. Then Rxy asserts that the notion of an animal is used in the definition of the notion of an animal with a horse's head and a horse's body. This relationship is depicted in Figure 11 by the directed arc leading from the node representing x to that representing y .

Notice that in this example, the same GCN net depicted in Figure 11 would emerge if R were the relation defined by asking "what must one know before one can learn ____?" For example, in terms of the available Primary Notions one could ask, "What must one know before one can learn about animals with horse's heads and bodies?" The response that emerges is "animals" and "horses." Thus, the identical structure would result.

GCN Example 2

A single GCN relation was used in the unicorn example. An example of a multiple relational GCN is needed to: illustrate a more complex GCN net, introduce graph and matrix representations of the GCN relations, illustrate an extension of the dependency relation, and act as a vehicle for discussing the task structure. A simple example based on an actual subject matter, of a multiple relational GCN cannot be given in a concise form. For this reason, we choose an imaginary subject matter with the following:

- 2 GCN relations, R, S
- 2 Primary Notions, $P1, P2$
- 2 Basic Principles, $B1, B2$
- 3 Secondary Notions, $S1, S2, S3$
- 4 Established Principles, $E1, E2, E3, E4$.

For concreteness, let the two GCN relations R and S have the meanings

Rxy for " x is used in formulating y "

Sxy for " x is used in establishing y "

where x and y are GCN constituents. These two relations are applicable to a variety of subject matters. The first relation is a semantic one and expresses the notion of what symbol, word, phrase, or sentence is used in stating another symbol, word, phrase, or sentence. Use of the word, "establishing," in the second relation, depending upon the subject matter, could mean the deductive, inductive, or experimentive arguments used to assert the GCN constituent y . The constituent x , in Sxy , is one of the constituents used in the argument.

We will describe the (hypothetical) structure imposed on the GCN by each of the relations starting with the relation R .

Consider first the Primary Notions. Table 2 summarizes, in matrix form, the hypothetical use of each of the Primary Notions in the formulation of each Basic Principle.

Table 2. Basic Principles Used in Formulating Primary Notions

		Basic Principles	
		B1	B2
Primary	P1	1	1
Notions	P2	1	1

An entry of "1" in the matrix (Table 2), indicates the Primary Notion in that row is used in the formulation of the Basic Principle heading the marked column. The matrix can be depicted as a graph as shown in Figure 12. In the graph, hexagonal nodes depict Primary Notions, and circular nodes depict Basic Principles. Directed lines, joining pairs of the nodes, signify the presence of the relation "used in the formulation of," between the Primary Notions and Basic Principles.

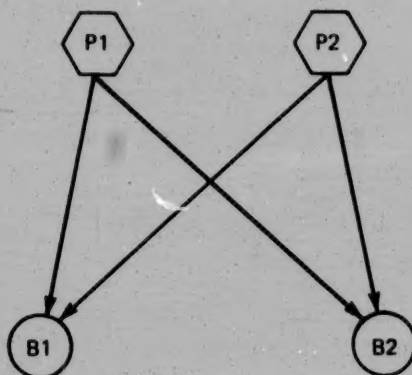


Figure 12. Graphic representation of the matrix in Table 2.

Primary Notions are also used hypothetically to formulate Secondary Notions as shown in Table 3. The Secondary Notions, S1 and S2, are formulated solely in terms of Primary Notions, but Secondary Notion S3 uses Secondary Notion S2 in its formulation. For completeness, the matrix in Table 3 also shows how Secondary Notions are used to formulate other Secondary Notions.

Table 3. Primary Notions and Secondary Notions Used in Formulating Secondary Notions

		Secondary Notions		
		S1	S2	S3
Primary Notions	P1	1	1	1
	P2	1	1	
Secondary Notions	S1			
	S2			1
	S3			

Figure 13 presents this portion of the relation R graphically. Secondary Notions are enclosed in triangles in Figure 13.

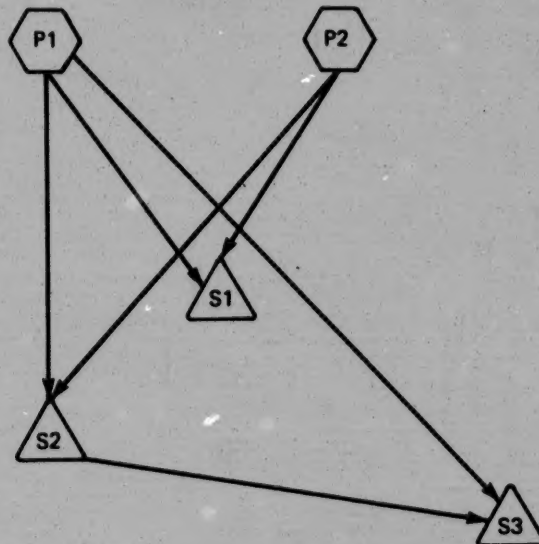


Figure 13. Graphic representation of the matrix in Table 3.

The final portion of R that needs to be depicted concerns the use of Primary and Secondary Notions in the formulation of the Established Principles. This has been done with the matrix shown in Table 4.

Table 4. Primary and Secondary Notions Used to Formulate Established Principles

		Established Principles			
		E1	E2	E3	E4
Primary Notions	P1	1	1	1	1
	P2	1	1	1	
Secondary Notions	S1			1	1
	S2			1	
	S3				1

The three matrices given in Tables 2, 3, and 4 can be combined to form the single matrix shown in Table 5. The rows and columns of the matrix in Table 5 are identical

and the matrix, using graph theoretical terminology, is called the adjacency matrix of the relation R and will be designated by $A(R)$.

Table 5. Adjacency Matrix $A(R)$ of the Relation R

	P1	P2	B1	B2	S1	S2	S3	E1	E2	E3	E4
P1			1	1	1	1	1	1	1	1	1
P2			1	1	1	1		1	1	1	
B1											
B2											
S1										1	1
S2							1			1	
S3											1
E1											
E2											
E3											
E4											

Figure 14 is the graph representation of the relation R , "used in the formulation of," and is obtained from the adjacency matrix $A(R)$ of the relation given in Table 5.

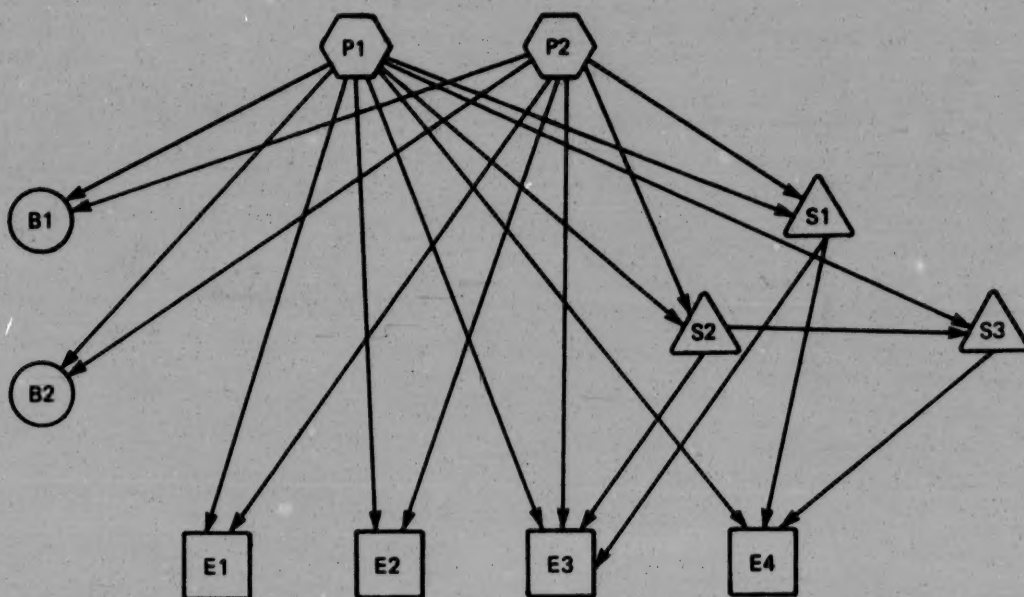


Figure 14. Graphic representation of the relation R .
(From the adjacency matrix of Table 5).

Figure 14 summarizes the information presented in Tables 2 thru 5 and Figures 12 and 13, and, as such, provides a complete graphic representation of the "used in the formulation of" relationship.

Figure 14 is the customary way to draw a graph of a relation. Such a representation rapidly becomes confusing to interpret with increasing numbers of GCN constituents. For this reason, an alternative graphic representation of the relation *R* is shown in Figure 15. In this figure, a dot at the intersection of two lines indicates the two lines are connected. The absence of a dot indicates the lines merely cross and are not connected. To illustrate the interpretation of the graph in Figure 15, consider the Primary Notion P1. Following the line drawn from P1 leads to a three-way connection. The left hand branch leads to the Basic Principles and following the connected lines leads to B1 and B2. Tracing the right hand branch below P1 leads to S1, S2, and S3. Tracing the bottom line leads to E1, E2, E3, and E4. The far right of Figure 15 shows how the Secondary Notions feed down into the Established Principles. For example, S1 is used in the formulation of E3 and E4.

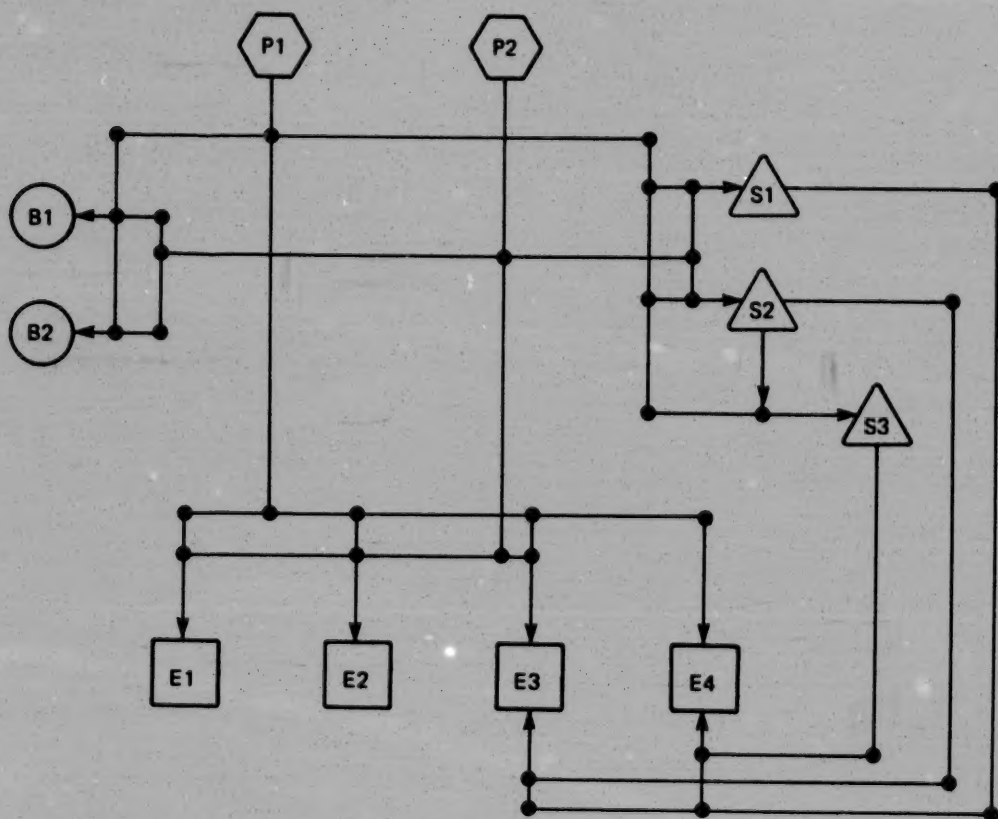


Figure 15. Alternative representation of the graph of the relation *R*.

Now consider the relation Sxy , "x is used to establish y." In the expression, Sxy , only names of constituents that are Established Principles can be substituted for y while x can be the name of any constituent (Primary Notion, Secondary Notion, Basic Principle, Established Principle). Since our interest is in structure, our concern is with the antecedents used in an Established Principle rather than with how the antecedents are actually used in establishing the principle.

Suppose the adjacency matrix, $A(S)$, of the relation S , of our hypothetical subject matter, is the one shown in Table 6. The rows and columns of this matrix are identical to those of the adjacency matrix of relation R of Figure 15. A column labeled by an Established Principle shows, by 1s in the row, the Secondary Notions, Basic Principles, and Established Principles used in arriving at the particular Established Principle. The graph of relation S is shown in Figure 16.

Table 6. Adjacency Matrix $A(S)$ of the Relation S

	P1	P2	B1	B2	S1	S2	S3	E1	E2	E3	E4
P1								1	1	1	1
P2								1	1	1	1
B1								1	1	1	
B2											1
S1										1	1
S2										1	
S3											1
E1									1		1
E2											
E3											1
E4											

The value of the relations R and S in the preceding hypothetical subject matter is that they provide the basis on which the instructional process for the subject matter must be constrained. For example, consider E3. As shown by Table 5 (or by Figure 15), P1, P2, S1, and S2 are all used in the formulation of E3. Thus, a student must have an understanding of these four constituents before he can comprehend the statement of E3. Further, from Table 6 (or Figure 16), since E3 is established by using P1, P2, B1, S1 and S2, the student must understand these constituents before the particular argument used to establish E3 can be understood. Note that B1 is used in the establishment of E3, but it was not used in formulating E3. Hence, relation R is not sufficient by itself to construct the instructional precedences for the GCN constituents. The precedence of B1 becomes evident only after a consideration of relation S .

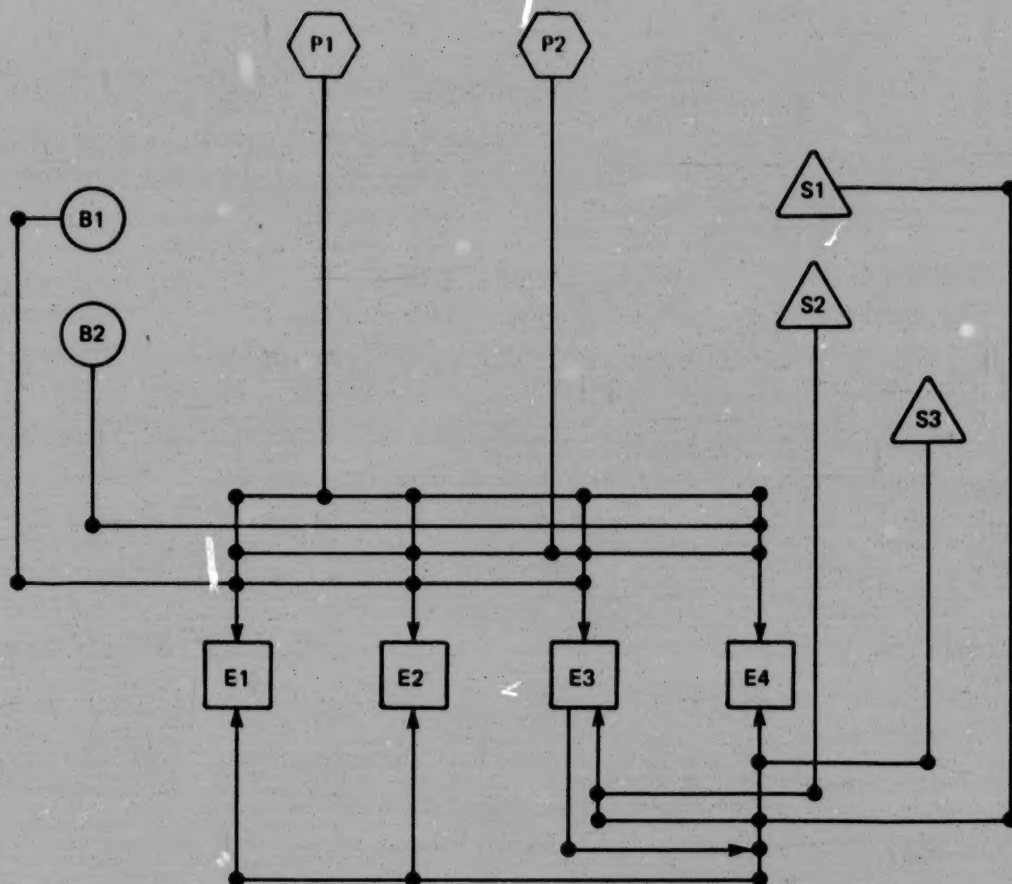


Figure 16. Graphic representation of the adjacency matrix of the relation S.

It is not our purpose, in this chapter, to develop the concepts of precedence, and so forth, that are used informally in the preceding discussion. These formal properties will be developed subsequently. Suffice it to say that precedence can be defined in terms of the notion of dependence introduced earlier. Recall, from Axiom 6, that it can be said of two constituents x, y that y depends on x if there are GCN constituents $x_1, \dots, x_n$ and GCN relations $R_1, R_2, \dots, R_{n+1}$, such that the following holds:

$$R_1 x x_1 \ \& \ R_2 x_1 x_2 \ \& \ \dots \ \& \ R_{n+1} x_n y$$

The GCN relations for a particular subject matter can be combined to produce the dependency relationship characterized in Axiom 6. The general method for doing this will be described next, after which these methods will be applied to develop the dependency relation for the hypothetical example just described.

DEPENDENCY

Every GCN relation has been shown to be irreflexive (Theorem 2i) and to have no loops (Theorem 2ii), and it is clear that the graph of any GCN relation does not contain parallel arcs. Thus, a graph of a GCN relation is a *digraph*. (Harary *et al.* 13). For example, the graphs in Figures 12, 13, 14, 15, and 16 are all digraphs. Earlier, we used the idea of the adjacency matrix of a relation, but did not formally define it. Formally then, let R be a GCN relation, and let there be n constituents in the GCN. The *adjacency matrix* for R is defined as the $n \times n$ square matrix $A(R) = [a_{ij}]$, where $a_{ij} = 1$ if constituent i is related to constituent j by R , and $a_{ij} = 0$ otherwise.

In what follows, the notion of Boolean arithmetic is needed. The only difference between Boolean and non-Boolean arithmetic is that in Boolean addition $1 + 1 = 1$. Following Harary *et al.* (13), we denote Boolean addition by writing $(n + m)\#$. So that

$$(1 + 1)\# = 2\# = 1,$$

and, in general, it follows that if $n > 1$, then $n\# = 1$.

Definition 3. Let a GCN have relations $R_1, R_2, \dots, R_n$ and n constituents. The adjacency matrix $A(D)$ for the dependency relation for the GCN is defined as $A(D) = [A(R_1) + A(R_2) + \dots + A(R_n)]\#$.

According to Definition 3, to form the adjacency matrix for the dependency relation in a GCN, the adjacency matrix for each GCN relation is constructed first. These adjacency matrices are then added element by element by using Boolean addition. This is normal matrix addition, except that the individual sums are Boolean sums, so that each entry in $A(D)$ is either a 1 or a 0. It follows that each element a_{ij} of $A(D)$ is such that:

$a_{ij} = 1$, if constituent i is related to constituent j , by *at least one* GCN relation and

$a_{ij} = 0$, if there is no GCN relation that relates constituent i to constituent j .

From this it can be seen that $A(D)$ depicts the dependency relationship to the extent of showing all dependency chains of length 1. The matrix $A(D)$ does not necessarily depict *all* the dependency relations, since dependency may be established through relational chains of length greater than 1. The next theorem provides the rationale needed to generate the complete dependency matrix.

Theorem 5. (Harary *et al.*, 13). Let $A(D)$ be the adjacency matrix for the dependency relation in a GCN. Then, in $A^n(D)$, [$A(D)$ raised to the n^{th} power], the i, j^{th} entry is the number of GCN relational sequences of length n leading from constituent i to constituent j .

It can be shown that for some $m > 1$, $A^{m+1}(D) = [0]$. This follows since, as shown in Theorem 3, there are no loops in the dependency relation. Since the number of

constituents is finite, there is an upper limit on the length of possible relational sequences leading from one constituent to another. This upper limit is denoted by m and is called the *maximal length dependency chain* for the GCN. It follows that $A^{m+1}(D) = [0]$, or the matrix containing only entries of 0.

In raising the dependency matrix to successive powers, ordinary row-by-column matrix multiplication is used. However, we are also interested in the matrix that results when the element-by-element products are summed by Boolean arithmetic.

Definition 4. The total dependency matrix, $A^*(D)$, for a GCN is defined as $A^*(D) = [A(D) + A^2(D) + \dots + A^m(D)] \#$, where m is the maximal length dependency chain for the GCN.

Suppose $A^*(D)$ is the total dependency matrix for a GCN, and let x and y be GCN constituents. Within $A^*(D)$, x will correspond to some row and y will correspond to some column. We will write a^*_{xy} to indicate the entry in $A^*(D)$ corresponding to the x^{th} row and y^{th} column.

The above results can be summarized by:

Theorem 6. Let x, y be GCN constituents, then y is dependent on x if and only if $a^*_{xy} = 1$.

Another useful result is obtained when ordinary arithmetic, instead of Boolean arithmetic, is used in forming the powers of the adjacency matrix $A(D)$. Let $\hat{A}^p(D)$ represent the matrix obtained by raising $A(D)$ to the p^{th} power using ordinary arithmetic. We then have the following theorem (Hammer and Rudeanu, 14):

Theorem 7. The element $a_{ij}^{(p)}$ of the p^{th} ordinary power of the adjacency matrix $A(D)$ is equal to the number of paths of length p joining i to j .

Theorem 7 will be illustrated later with an example.

Thus, Theorems 6 and 7, and the concepts on which they are based, provide a direct method of developing the dependency relationship from the individual GCN relations. From an instructional point of view, the dependency relationship is crucial to the sequencing of instructional material. Instruction cannot be given concerning a constituent in a subject matter GCN until the student is known to understand or comprehend all constituents upon which the first constituent depends. In other words, if a constituent, c , depends on each element in a set of constituents, γ , then instruction concerning c cannot proceed until the instructional agent knows or can assume that an individual understands all elements in γ . In this sense, γ consists of all constituents that precede c . To summarize, the dependency relation constrains the order in which the instructional agent can schedule instruction in a particular subject matter.

Dependency: Continuation of the Hypothetical Example 2.

The hypothetical example developed earlier will be extended in order to provide an example of the process of formulating the dependency relationship. The example contained two GCN relations: R , "used in the formulation of," and S , "used in the proof of." Those relations are depicted graphically in Figures 15 and 16, respectively, and the adjacency matrices for these relations are given in Tables 5 and 6, respectively. The matrices in Tables 5 and 6 are combined by Boolean arithmetic to produce the adjacency matrix, $A(D)$, for the dependency relation in the hypothetical GCN as shown in Table 7. Figure 17 is the graph representation of the adjacency matrix $A(D)$ of Table 7.

Table 7. Adjacency Matrix, $\underline{A(D)} = [\underline{A(R)} + \underline{A(S)}] \#$,
for Dependency Relation $\underline{D}$

	P1	P2	B1	B2	S1	S2	S3	E1	E2	E3	E4
P1			1	1	1	1	1	1	1	1	1
P2			1	1	1	1		1	1	1	1
B1								1	1	1	1
B2											1
S1										1	1
S2							1			1	
S3											1
E1									1		1
E2											
E3											1
E4											

Tables 8 and 9 give the dependency matrices of $A^2(D)$ and $A^3(D)$, respectively. The dependency matrix $A^4(D)$ is the null matrix, and hence the longest possible dependency chain is of length 3. If an element in $A^2(D)$ is equal to 1, then there exists at least one path of length 2 from the row element to the column element. Thus, in $A^2(D)$, the (P2,S3) element is equal to 1 and one or more paths go from P2 to S3. Referring to Figure 17, one such path is (P2,S2,S3). If an element in $A^3(D)$ is equal to 1, then there exists at least one path of length 3 from the row element to the column element. For example, in $A^3(D)$, the (P1,E4) entry is equal to 1. Referring to Figure 17, (P1,B1,E3,E4) is one such path. Another path of length 3 is (P1,S2,S3,E4).

Table 10 is the total dependency matrix $A^*(D)$. An entry of 1 in a particular row and column of $A^*(D)$ means that at least one chain leads from the particular row

Table 9. Dependency Matrix $\underline{A}^3(D)^\#$

	P1	P2	B1	B2	S1	S2	S3	E1	E2	E3	E4
P1											
P2											
B1									1		1
B2									1		1
S1											
S2											
S3											
E1											
E2											
E3											
E4											

element to the particular column element. Inspection of $A^*(D)$ does not tell the length of the chain leading from one constituent to another; it merely tells of the existence of a chain.

Table 10. Dependency Matrix
 $\underline{A}^*(D) = [\underline{A}(D) + \underline{A}^2(D) + \underline{A}^3(D)]^\#$

	P1	P2	B1	B2	S1	S2	S3	E1	E2	E3	E4
P1			1	1	1	1	1	1	1	1	1
P2			1	1	1	1	1	1	1	1	1
B1								1	1	1	1
B2											1
S1										1	1
S2							1			1	1
S3											1
E1									1		1
E2											
E3											1
E4											

Tables 11 and 12 show matrices $\hat{A}^2(D)$ and $\hat{A}^3(D)$, respectively, for the hypothetical example with its adjacency matrix $A(D)$ as given in Table 7. We can verify the entries in $\hat{A}^3(D)$ by actually counting the paths in the net representation of $A(D)$ given in Figure 17. The results are shown in Table 13.

Table 11. The Matrix $\hat{A}^2(D)$ Giving the Number of Paths of Exactly Length 2

	P1	P2	B1	B2	S1	S2	S3	E1	E2	E3	E4
P1								1	2	3	6
P2							1	1	1	1	1
B1									1		1
B2											
S1											1
S2											1
S3											
E1											
E2											
E3											
E4											

Table 12. The Matrix $\hat{A}^3(D)$ Giving the Number of Paths of Exactly Length 3

	P1	P2	B1	B2	S1	S2	S3	E1	E2	E3	E4
P1									1		3
P2									1		3
B1											
B2											
S1											
S2											
S3											
E1											
E2											
E3											
E4											

Table 13. Number of Paths of Length 3 as Taken From Figure 17

From $\hat{A}^3(D)$ (Table 12)			From Figure 17	
No. Paths	From	To	Paths	
1	P1	E2	P1, S1, E1, E2	
3	P1	E4	P1, B1, E1, E4 P1, S2, E3, E4 P1, S2, S3, E4	
1	P2	E2	P2, B1, E1, E2	
3	P2	E4	P2, B1, E1, E4 P2, S2, E3, E4 P2, S2, S3, E4	

TASKS IN CONTEXT STRUCTURE

The description of the GCN portion of the subject matter structure is complete and the next problem discussed is that of developing the theoretical framework for the Task in Context Structure, TICS.

The idea underlying the TICS is closely related to that of a criterion test. During instruction, some means must be available to determine whether or not the student understands the material being presented. This is usually done by periodically presenting the student with a set of tasks to perform. A judgment concerning student comprehension is then made on the basis of observations of student behavior. Observable behavior consists of responses to questions, information supplied by the student, and so forth.

This process is used for all GCN constituents. In particular, understanding of Primary and Secondary Notions can be tested by having students apply the notions. Basic Principles can be considered as characterizing the Primary Notions, and students can be tested concerning their understanding of Basic Principles by having them interpret, reformulate, or apply these constituents. Student comprehension of Established Principles can be tested by having the student interpret, apply, or prove these principles.

In general, then, there is a set of tasks such that task performance can be observed behaviorally. Each GCN constituent is associated with *at least one subset* of the tasks. Performance on all tasks within one of the task subsets associated with a constituent is assumed to be sufficient grounds to infer comprehension of the GCN constituent. More than one task subset can be associated with one constituent since the criterion subset presented to a student will depend on the individual student and his instructional history.

For example, consider a student receiving instruction on a particular constituent. One task subset may be presented to the student, and it may result that his performance is inadequate and, thus, leads to remedial instruction. After remedial instruction, it may happen that retesting on comprehension of the same constituent is desirable. In this case, it would seem that a new set of criterion tasks would be appropriate, so that successful performance cannot be attributed to prior exposure rather than comprehension. Thus, in general, each constituent will be associated with several task subsets.

It is important to notice that performance on a particular task subset is only sufficient (not necessary) grounds for inferring comprehension of the constituent. That is, from constituent comprehension, it is not possible to infer performance on a *particular*

task subset. (If there is only one task subset associated with a constituent, comprehension infers performance ability on those tasks.) It is only possible to infer performance ability on at least one of possibly many task subset(s) assuming constituent comprehension.

These points are explicitly formalized in the following axioms that characterize the TICS.

Axiom 7. There is a finite set of tasks associated with the GCN such that it can be determined by behavioral observation for each task whether or not an individual can perform the task.

Axiom 8. Every GCN constituent c is associated with a unique collection of subsets of the set of tasks.

On the basis of these axioms, we can introduce some formal notions to simplify the succeeding discussion. First, we introduce a means of referring set theoretically to the set of tasks.

Definition 5. Let

$$T = \{t_1, t_2, \dots, t_n\}$$

designate the set of tasks associated with the GCN.

Axiom 8 asserts the existence of a relation from any single element c in the GCN to a unique set of subsets of the set of tasks. The phrase, "the set of subsets of a set," occurs so often in mathematics that it has been given a special designation, namely, "the power set of a set." In short, the power set of a given set is the set of all the subsets of the given set. To illustrate, consider the set of three tasks, $T = \{t_1, t_2, t_3\}$. Then the power set of T , denoted by $P(T)$, is the following set:

$$P(T) = \{\phi, \{t_1\}, \{t_2\}, \{t_3\}, \{t_1, t_2\}, \{t_1, t_3\}, \{t_2, t_3\}, \{t_1, t_2, t_3\}\},$$

where ϕ indicates the empty set. Notice that the elements of $P(T)$ are sets.

It will be convenient to have a condensed notation for the elements of the power set. We let T_i denote an arbitrary element in $P(T)$. If T contains n tasks, then there are 2^n subsets in $P(T)$ so that $1 \leq i \leq 2^n$. The order of the indices used to label the elements of $P(T)$ is not important. To illustrate the labeling process, if T is the three-task set example, one possible labeling of the elements of $P(T)$ is:

$$\begin{aligned} T_1 &= \phi \\ T_2 &= \{t_1\} \\ T_3 &= \{t_2\} \\ T_4 &= \{t_3\} \\ T_5 &= \{t_1, t_2\} \end{aligned}$$

$$T_6 = \{t_1, t_3\}$$

$$T_7 = \{t_2, t_3\}$$

$$T_8 = \{t_1, t_2, t_3\}$$

To summarize, in more general terms, suppose the set T consists of n tasks so that

$$T = \{t_1, t_2, \dots, t_n\}.$$

Then the power set of T will have 2^n elements, each element being a unique subset of T .

Thus, one can write

$$P(T) = \{T_1, T_2, \dots, T_k\},$$

where $k = 2^n$. We will refer to the elements of $P(T)$, the $T_1, T_2, \dots, T_k$, as task sets.

Definition 6. If c is a GCN constituent, then F_c denotes the set of task sets associated with c .

Generically, F is a relation from the GCN to a subset of the power set of T , $P(T)$.

In more detail, if T_{i_j} denotes an element of $P(T)$, then

$$\{T_{i_1}, T_{i_2}, \dots, T_{i_m}\}$$

represents a subset of the power set. With this notation, if c is a particular GCN constituent, then F_c can be expressed as

$$F_c = \{T_{i_1}^c, T_{i_2}^c, \dots, T_{i_m}^c\}$$

In the above, the superscript on the task sets is not an ordinary exponent, but indicates the association of the task sets with a particular GCN constituent c .

Figure 18 is an illustration of a possible relation F_c for our simple example of a set of three tasks.

In the illustration,

$$F_c = \{T_2^c, T_3^c, T_7^c\},$$

where:

$$T_2^c = \{t_2\}, T_3^c = \{t_3\} \text{ and } T_7^c = \{t_2, t_3\}.$$

To summarize, the relation F from the set of constituents into the power set of the set of tasks provides the coordinating relation referred to in the Introduction. This relation serves to relate the components of the GCN, constituents, to components of the TICS, collections of sets of tasks. On the basis of this relation, the dependency structure imposed on the GCN by the GCN relations can be seen to carry over to the TICS. Consider two constituents x, y such that y depends on x . Each of these constituents is associated with a particular class of task sets in the TICS. Analogous to the GCN dependency relation, the task sets associated with x can be seen to precede the task sets associated with y . In fact, as will be seen below, a student will not be able to perform all

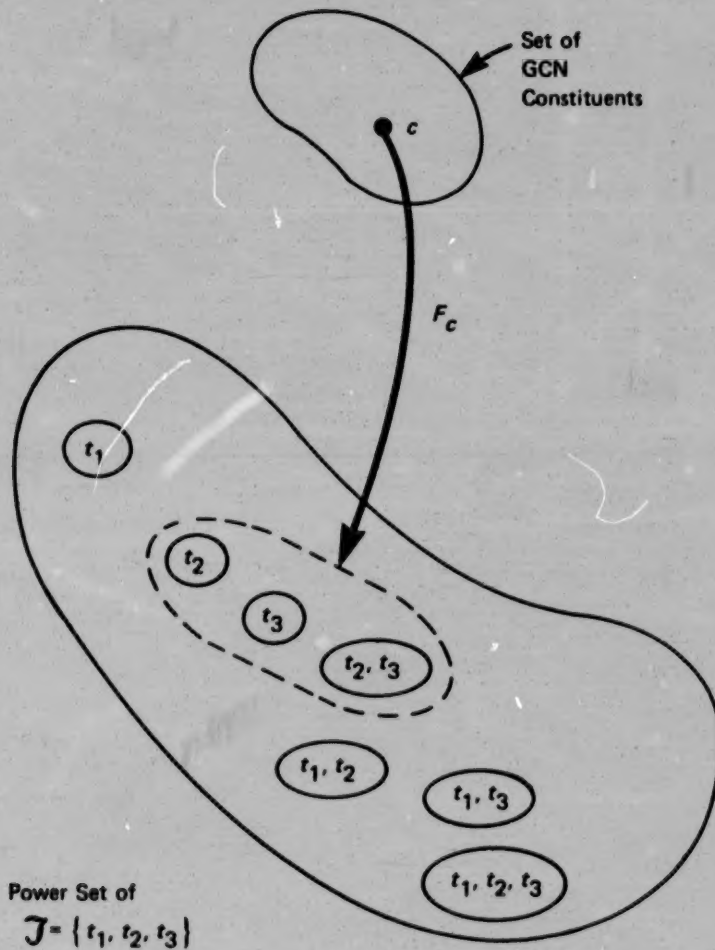


Figure 18. Illustration of a possible relation F_c for a set of three tasks.

tasks in one task set in F_y until he can perform all tasks in at least one task set in F_x . This result follows on the basis of the dependency of y on x , as well as the existence of the coordinating relation F .

The TICS axioms make use of the primitive instructional concepts of tasks, task performance, and criterions associated with a constituent. On the basis of these concepts, we introduce the notion of comprehension of a constituent.

Definition 7. Let c be a GCN constituent. An individual is said to comprehend c if and only if there is at least one task set, say T_i^c , in F_c such that the individual can perform all the tasks in T_i^c .

Task sets of F_c can be of equal or unequal difficulty. All Definition 7 asserts is that an individual has to perform all the tasks in at least one of the task sets before it can be said that the individual comprehends c . The definition does not rule out the possibility of an individual being able to perform all the tasks in more than one task set.

To illustrate Definition 7 further, let $\{t_8, t_9, \dots, t_{21}\}$ be the subset of the set of tasks associated with a GCN constituent c of some subject matter. Suppose the tasks can be grouped into the following four task sets:

$$T_1^c = \{t_8, t_9, t_{11}, t_{15}, t_{18}, t_{20}\}$$

$$T_2^c = \{t_8, t_9, t_{13}, t_{15}, t_{19}\}$$

$$T_3^c = \{t_8, t_{10}, t_{11}, t_{16}\}$$

$$T_4^c = \{t_8, t_{12}, t_{14}, t_{17}, t_{21}\}$$

Thus, the set of task sets for c is

$$F_c = \{T_1^c, T_2^c, T_3^c, T_4^c\}.$$

Notice that, as in the choice of GCN constituents and relations, there can be a great deal of latitude in the selection of tasks to form task sets. This appears reasonable because the individual preparing the instructional material is free to choose and formulate tasks as desired.

In the above example, if an individual can perform all tasks in one of the T_1^c, T_2^c, T_3^c or T_4^c , then it can be said that the individual comprehends the notion represented by c .

The final axiom to be introduced establishes a structure within the TICS that is based on the structure imposed on the GCN by the GCN relations.

Axiom 9. Let x and y be GCN constituents, R a GCN relation, and assume Rxy . If there is a task set T_i^y in F_y , such that an individual can perform all the tasks in T_i^y , then there is a task set T_j^x in F_x such that the individual can perform all tasks in T_j^x .

To illustrate Axiom 9, suppose Rxy means, "x is used in establishing y." Let

$$F_y = \{T_1^y, T_2^y, \dots, T_i^y, \dots\}$$

designate the collection of task sets associated with GCN constituent y . Let

$$F_x = \{T_1^x, T_2^x, \dots, T_j^x, \dots\}$$

denote the collection of task sets associated with GCN constituent x . Suppose an individual can perform all the tasks in one of the task sets, say T_i^y , of F_y . Then, because x is used in establishing y , the implication is that the individual can perform all the tasks in some task set, say T_j^x , of F_x .

On the basis of Axiom 9, we can establish the following theorem asserting that for an individual to comprehend a constituent, all constituents on which it depends must be comprehended.

Theorem 8. If an individual comprehends a constituent y , then the individual must comprehend all constituents on which y depends.

Notice that Theorem 8 cannot be strengthened to an if-and-only-if condition. That is, because an individual comprehends all the constituents that a constituent x depends on, it does not necessarily follow that the individual comprehends x . This is intuitively correct, since the purpose of instruction is to permit the individual to synthesize the knowledge he possesses in order to develop more understanding. That is, when an individual does comprehend all the constituents x depends on, then and only then is the individual prepared to go on and receive instruction on x . As a result of this instruction, the individual is expected to develop comprehension concerning x .

This completes the description of the TICS. The TICS together with the GCN and the coordinating relations form the *Subject Matter Structure* for the particular subject matter under consideration. To summarize, the GCN constituents are structured through one or more GCN relations. Those relations can be combined as described above to form the dependency relation for the GCN. Underlying the GCN structure is the TICS. The dependency structure in the GCN imposes a structure on the task subsets through the coordinating relation. Judging constituent comprehension is based upon the performance on associated criterion sets. Satisfactory performance implies constituent comprehension, which, in turn, implies satisfactory performance on at least one criterion set associated with each constituent depended upon.

As will be seen in the next chapter, this descriptive theory of subject matter structure can be used in conjunction with student state descriptors to impose constraints on instructional sequencing.

Chapter 2

A REPRESENTATION OF STUDENT STATES AND THE GOAL OF INSTRUCTION

In this chapter, the focus is on Pask's second, third, and fourth components. In particular, a theory is developed that provides a representation of the student state at any time: either the initial state, some subsequent intermediate state, or the terminal state. A representation of the instructional goal is also developed. The theory developed herein can be viewed as an extension of the axiomatic subject-matter structure theory developed in Chapter 1. That is, further axioms will be introduced that provide for student state descriptions. Thus, a unified axiomatic theory will result in which both the subject matter and the student state can be represented.

This result is precisely what is required when the problem of providing a model of instruction is considered. The collection of student state descriptions, which will be discussed, can be seen as the totality of possible student states. As a result of instruction, the student moves from state to state. However, the subject-matter structure will be seen to constrain the possible student-state transitions. More precisely, the student-state description includes information concerning the students' mastery of subject-matter components (constituents). However, the subject-matter constituents are hierarchically ordered by a dependency relationship. Possible state transitions are constrained because the state descriptors contain information relating to the mastery of constituents that are known to be ordered by the dependency relationship. The details of these interrelationships are discussed in this chapter. In the first section of this chapter, a student-state description is developed. In the second section, a definition of the goal of instruction is presented.

STUDENT STATE DESCRIPTIONS

The purpose of this section is to develop a mechanism that will provide for the representation of the state of the student at any point in time (initial state, current state,

terminal state). What is needed is a formal way of describing the qualities that the student possesses. In other words, we are concerned with the *attributes* of the student. In this sense, we need to be able to describe the attributes that the student possesses and those that he does not possess.

The interest is not in every attribute that can possibly be formulated, because there are an infinite number of possible attributes; but we are interested in those attributes that an instructor deems important from an instructional point of view for a given subject matter to be taught within a given context. As discussed in Chapter 1, the context of the instruction serves to delimit the scope of the subject matter and the behavioral objectives (tasks). In a similar way, the context of the instruction serves to delimit the set of relevant student attributes.

Formally, we have the following assumption.

Axiom 10. There is a finite set of attributes such that an instructor can express all that he is interested in, as regards any student, by means of these attributes. Furthermore, it can be determined for each attribute whether or not a student possesses that attribute.

The second portion of this axiom is essential, since an attribute would be worthless from an instructional point of view, unless it were possible to determine whether or not an individual possesses the attribute.

As described earlier, the GCN contains a collection of subject-matter constituents. Since the instructor has identified these as the concepts, notions, and so forth, comprising the subject matter, it is reasonable to assume that he is interested in whether or not an individual comprehends each of these constituents. That is, for each constituent in the GCN, we expect the collection of attributes to contain an attribute associated with that constituent. In addition, it is assumed that an individual has the attribute associated with a constituent, just in the case the individual comprehends that constituent. Axiom 11 summarizes those remarks, where A is the set of attributes referred to in Axiom 10.

Axiom 11. For every GCN constituent, c , there is an attribute $a_c \in A$ such that an individual has attribute a_c if and only if the individual comprehends c .

Denote the number of GCN constituents by m , and suppose the set A contains a total of n attributes; then $m \leq n$, and write

$$A = \{a_1, a_2, \dots, a_m, a_{m+1}, \dots, a_n\}$$

to denote the set of elements constituting A . Thus, $a_1, a_2, \dots, a_m$ are the m attributes associated with the m GCN constituents. The index, i , on a_i , where $i = 1, 2, \dots, m$, refers to the m GCN constituents. Furthermore, $a_{m+1}, a_{m+2}, \dots, a_n$ are the $(n-m)$ attributes

associated with the $(n-m)$ non-GCN attributes. In this case, the index j on a_j , where $j = m+1, \dots, n$, refers to the non-GCN attributes.

Beyond the m attributes in A associated with the GCN constituents, the instructor is free to choose any attributes he deems relevant to the instruction. This parallels the freedom of choice available in choosing particular Primary Notions, Secondary Notions, Basic Principles, and Established Principles. As argued earlier, it is simply not possible to specify one particular set of GCN constituents as being *the* correct set of constituents for a particular subject matter. What constituents are chosen and the manner in which they are structured depends upon the context, the use to be made of the knowledge that is to be learned, the personal preference on the instructor's part, and so forth. Similar remarks apply to the selection of attributes. Examples of attributes, which are not GCN related, are attributes such as IQ, level of aspiration, score on a particular test battery, age, number of years in education, and reading proficiency.

General guidelines concerning the selection of non-GCN related attributes may be possible; however, it does not seem reasonable to expect to be able to provide an exhaustive set of attributes suitable for a given subject matter for any context. In fact, one interesting question concerns what attributes are important for various subject matters in various contexts. For example, Seidel (15) reports that level of aspiration is perhaps an attribute that has important instructional ramifications in some contexts. No pretext is made concerning the solution to this problem. It suffices for the developments to be given that it is simply assumed that a set of relevant attributes has been chosen. This set of attributes includes one associated with each GCN constituent (Axiom 11). Furthermore, for each attribute, it can be determined whether or not an individual possesses that attribute (Axiom 10).

As will be argued, the purpose of instruction is to modify the student's attributes. In other words, the purpose of instruction is to change the qualities that the student possesses. In particular, as instruction proceeds, it is expected that the student will begin to comprehend more and more of the GCN constituents. Thus, more attributes associated with the GCN constituents are attributable to the student as instruction progresses.

The $(n-m)$ attributes not associated with GCN constituents are not of this type. While these attributes will vary over the course of instruction, the view maintained herein is that these changes are not the goal of the instructional process. Consider, for example, an attribute such as understanding the definition of a word. Instructionally, it is an important question whether or not an individual understands the definition of a word, if it is used in the course of instruction. However, if the definition of a word is part of the

instructional process, then it must be considered as being part of the subject matter, hence, the GCN itself. Therefore, the goal of instruction does not include modification of student qualities as regards these $(n-m)$ attributes.

Given the set of attributes, it remains to provide a manner of formally asserting whether or not an individual has an attribute. Predicate Logic provides a natural mechanism for asserting that an individual has or does not have some attribute (e.g., Suppes, 10). The accepted convention is to write, for example, Px to assert that individual denoted by x has the attribute denoted by P . If P stands for the attribute "has a high school education," then Px asserts that x has a high school education. We thus introduce n predicates $P_1, P_2, \dots, P_n$ to reflect the n attributes of the attribute set A . The attributes in A are exhaustive in an instructional sense; thus, the predicates represent everything deemed essential about the student from an instructional point of view. (Lower case English letters, $x, y, \dots$, will be used to denote arbitrary individuals. The letters $x, y, \dots$ will be referred to occasionally as individual names.)

Predicates and individuals are combined by the following definition:

Definition 8. An atomic formula consists of one predicate followed by one individual name. Thus, $P_i x$ and $P_i y$ are both atomic formulas. From Definition 8, $P_i x$ holds if and only if individual x has attribute i , where $i = 1, 2, \dots, n$. In addition, $\neg P_i x$ holds if and only if individual x does not have attribute i , where $i = 1, 2, \dots, n$. Notice, in particular, $P_i x$, where $i = 1, 2, \dots, m$ asserts individual x comprehends GCN constituent i .

The n predicates $P_1, \dots, P_n$ can be used to develop the individual state descriptions that will be used to represent the state of the student. The formal basis for this development is drawn from the theory of Inductive Logic of Carnap (16, 17). The required definitions are given below with a detailed example.

Definition 9. An S_i predicate is a lexicographically ordered conjunction of atomic formulas in which every predicate appears exactly once either negated or unnegated.

By way of an example, suppose the following three predicates:

P_1 : comprehends constituent c_1 ,

P_2 : comprehends constituent c_2 ,

P_3 : has an IQ greater than 100.

Then, these three predicates, reflecting three attributes, lead to three atomic formulas for an arbitrary individual, x .

P_1x if and only if x comprehends constituent c_1

P_2x if and only if x comprehends constituent c_2

P_3x if and only if x has an IQ greater than 100.

Finally, we can construct eight S predicates from the three predicates:

$S_1x: P_1x \ \& \ P_2x \ \& \ P_3x$

$S_2x: P_1x \ \& \ P_2x \ \& \ \neg P_3x$

$S_3x: P_1x \ \& \ \neg P_2x \ \& \ P_3x$

$S_4x: P_1x \ \& \ \neg P_2x \ \& \ \neg P_3x$

$S_5x: \neg P_1x \ \& \ P_2x \ \& \ P_3x$

$S_6x: \neg P_1x \ \& \ P_2x \ \& \ \neg P_3x$

$S_7x: \neg P_1x \ \& \ \neg P_2x \ \& \ P_3x$

$S_8x: \neg P_1x \ \& \ \neg P_2x \ \& \ \neg P_3x$

In general, if there are n predicates, then there are 2^n S predicates. The S predicates are, in general, assumed to be ordered in some definite way. The set of S predicate will be designated as,

$$S = \{ S_1, S_2, \dots, S_k \}$$

where $k = 2^n$. The 2^n S predicates in S will be referred to as "states." The following discussion will make clear the reasons for this terminology.

Suppose S_i is an arbitrary S predicate, x an arbitrary individual, and each predicate in S_i is replaced by its associated atomic formula. That is, each occurrence of P_i or $\neg P_i$ in S_i is replaced by P_ix or $\neg P_ix$. Then, it can be said that individual x satisfies S predicates S_i .

Definition 10. Suppose S_ix holds for an arbitrary individual x ; in this case, this means x satisfies S predicate S_i and we say that x is in state S_i . An individual x , can be only in one state S_i . That is, if S_ix holds, then $\neg S_jx$ holds, for all $j \neq i$, where $i, j = 1, 2, \dots, 2^n$. The notation, S_ix , will be referred to also by saying that S_ix is the state description for the arbitrary individual, or student x .

As an illustration, in the example above, the eight S predicates associated with P_1, P_2 , and P_3 were listed and denoted by $S_1, S_2, \dots, S_8$. If x is an arbitrary student, then the eight student-state descriptions are as listed above.

Consider an individual x who comprehends c_1 , does not comprehend c_2 , and who has an IQ greater than 100. For this individual, it follows that

$$P_1x \ \& \ \neg P_2x \ \& \ P_3x$$

holds. Thus, x satisfies S_3 , and this is denoted by writing S_3x , and we say that x is in state S_3 .

THE GOAL OF INSTRUCTION

The elements of S provide a description of every possible student instructional state that the student can be in, and every student satisfies a unique S predicate in S . The set

$$S = \{S_1, S_2, \dots, S_k\}$$

where $k = 2^n$ of possible student states provides the formal mechanism through which it is possible to represent the initial, current, or terminal student state. When a student commences instruction, it must be determined which of the n attributes the student does and does not have. Once this has been done, the S predicate that the student satisfies can be identified. This leads to the assignment of the student to one of the states in S .

The goal of instruction can be seen as the alteration of the student's attributes structure. In particular, given GCN constituents $c_1, \dots, c_m$, it will be the case that the goal of instruction is to, at least, have the student comprehend each of $c_1, \dots, c_m$. Thus, as a result of instruction, the student would be expected to acquire the m attributes associated with $c_1, \dots, c_m$. As instruction proceeds, the student will move from state to state, with each state being a member of S . At the end of instruction, the student will be in some final terminal state, which is also a member of S . The final terminal state for any student will assert that the student has the m attributes associated with the m GCN constituents.

In discussing the goal of instruction, the only concern is with the GCN-related attributes. The $(n-m)$ non-GCN-related attributes may be important when providing instruction, but it is not the purpose of instruction to modify these attributes. As indicated above, the goal of instruction is to place the student in a state in which it is asserted that the student comprehends all GCN constituents.

It will be convenient to have a device through which one can refer to the subject-matter-related attributes affirmed in an arbitrary state description. The idea here is that each student state asserts the comprehension of a set (possibly empty) of GCN constituents. Associated with this set of constituents is a set of predicates appearing unnegated in the state description's respective S predicate. These unnegated predicates affirm in a positive way qualities that the student possesses that are subject-matter-related. These remarks are summarized in the following definition.

Definition 11. The set of all subject-matter predicates that are affirmed in state description S_i is denoted by $APRED(S_i)$, and represents the set

$$\{P_i : P_i \text{ is unnegated in } S_i \text{ and } 1 \leq i \leq m\}.$$

(In the above definition, PRED is an abbreviation for "predicate," and the letter A is an abbreviation for "affirmative.") The range of i , the subscript on P_i , is from 1 to m and indicates the predicates as all being subject-matter related. The set defined in Definition 11 will, for ease of expression, be called "the affirmative set of S_i ." When necessary, the phrase, "The subject matter affirmative set of state S_i ," will be used.

As an example of an affirmative set, suppose x satisfies state S_3 where S_3x is given by

$$S_3x: P_1x \ \& \ \neg P_2x \ \& \ P_3x$$

and that all three predicates in S_3 are subject-matter-related. Then the affirmative set of S_3 is

$$\text{APRED}(S_3) = \{P_1, P_3\}.$$

Definition 11 can be used to define precisely the goal of instruction. The goal of instruction is the set of state descriptions such that each of the state descriptions affirms comprehension of $c_1, \dots, c_m$. In other words, the goal of instruction is the subset of state descriptions such that each of the state descriptions contains each of the predicates associated with a constituent in unnegated form.

Definition 12. The goal of instruction, G , is defined as:

$$G = \{S_i: \text{APRED}(S_i) = \{P_1, P_2, \dots, P_m\}\}.$$

In other words, the goal of instruction, G , is a set. This set is the set of all states, S_i , whose affirmative set is the set $\{P_1, P_2, \dots, P_m\}$.

As an example, consider the three predicate examples used earlier in which there were two GCN constituents, c_1 and c_2 , and one non-GCN constituent, c_3 . Two student states assert comprehension of both c_1 and c_2 :

$$S_1x: P_1x \ \& \ P_2x \ \& \ P_3x$$

$$S_3x: P_1x \ \& \ P_2x \ \& \ \neg P_3x.$$

The subject-matter affirmative sets are then:

$$\text{APRED}(S_1) = \{P_1, P_2\}$$

$$\text{APRED}(S_2) = \{P_1, P_2\}$$

so that the goal of instruction is the set G given by $G = \{S_1, S_2\}$. Notice, as appropriate, the goal of instruction is neutral concerning the student's IQ, the third attribute. That is, the terminal student states are neutral with respect to the student's IQ—he can have an IQ either greater than 100, or less than or equal to 100. The goal of instruction is not to modify the student's IQ.

Therefore, we now have a mechanism for representing the student state at any point in time. Furthermore, we can also represent the goal of instruction as a set of terminal

student states. Finally, instruction can be viewed as a process that alters the student's GCN-related attributes. That is, instruction is a process by which the student progresses from state to state.

In the next chapter, this state transition concept of instruction will be formalized. Furthermore, the way in which subject-matter structure constrains student-state transitions, as mentioned earlier, will also be discussed.

Chapter 3

AN INSTRUCTIONAL MODEL

At this point in the development we have a formal representation of:

- (1) Subject-matter structure.
- (2) The student state: initial and current.
- (3) The goal of instruction: the set of terminal student states.

In this chapter, the subject-matter structure theory and the student state description theory are related in terms of an overall instructional model. Also investigated will be how the subject-matter structure constrains the student state transitions as the student passes from the initial state, through the current states to some terminal state. The subject-matter structure constrains the student-state transitions in two ways. First, not all student states need to be considered, since the GCN dependency relation can be used to eliminate the impossible student states from consideration. Second, the set of possible student states can then be ordered on the basis of the dependency relation to yield the collection of possible state-transition paths.

Consider first how the impossible student states can be eliminated. We assume the GCN contains m constituents $c_1, \dots, c_m$, and that they are associated with m predicates, $P_1, \dots, P_m$. Assume further that one of the constituents, c_j , depends on another, c_k . It follows that any state in which $\neg P_k \ \& \ P_j$ holds is an impossible state. That is, since c_j depends on c_k , comprehension of c_j requires comprehension of c_k . Thus, $\neg P_k \ \& \ P_j$ asserts both comprehension of c_j and non-comprehension of c_k , and this situation is contrary to Theorem 8 of Chapter 1. Thus, any state description asserting $\neg P_k \ \& \ P_j$ is known to be impossible.

The next theorem serves to provide the general basis on which impossible states can be eliminated. Two preliminary definitions are required. The first of these definitions serves to formalize the notions of impossible and possible student states, informally introduced above.

Definition 13. An *impossible* student state is a state description, S_i , such that there is an unnegated predicate, P_j , in S_i such that c_j (associated with P_j) depends on some c_k , but P_k (associated with c_k) is negated in S_i . A student state is *possible* if it is not impossible.

Thus, if constituent c_j depends on some constituent c_k , an impossible state description contains a predicate combination,

$$(i) P_j \& \neg P_k.$$

Possible student-state descriptions, on the other hand, can contain any one of the three predicate combinations:

$$(ii) P_j \& P_k$$

$$(iii) \neg P_j \& P_k$$

$$(iv) \neg P_j \& \neg P_k.$$

Notice that in discussing possible student states we need only be concerned with the m subject-matter-related predicates. That is, from the point of view of identifying possible states, and then sequencing these states, we need *not* concern ourselves with the $(n-m)$ nonsubject-matter predicates. As argued above, these attributes may be important when providing instruction, but the purpose of the instruction is not to modify these attributes. For example, reading ability may be an important consideration in the course of instruction. However, if the intent of instruction is to modify reading ability, then the reading-related attributes must be considered also as GCN-related attributes. In other words, the $(n-m)$ nonsubject-matter-related predicates play no role in restricting either the set of possible student states or possible sequencing of these states. The above definitions of impossible and possible student states restrict attention to GCN-related attributes as appropriate.

We have the following theorem to characterize possible student states.

Theorem 9. For all $i = 1, \dots, 2^n$, S_i is a possible student state if and only if constituent c_j depends on Constituent c_k , then $P_j \in \text{APRED}(S_i)$ implies $P_k \in \text{APRED}(S_i)$ where P_j is associated with c_j , P_k is associated with c_k and $j, k = 1, \dots, m$.

As an example of the way in which Theorem 9 can be applied, consider the example above based upon three attributes: comprehending c_1 , comprehending c_2 , and having an IQ greater than 100. In this example, only P_1 and P_2 are subject-matter-related. Assume, also, that c_1 depends on c_2 . From Theorem 9, we know that the only possible student states, S_i , are those for which the statement "if $P_1 \in \text{APRED}(S_i)$ then $P_2 \in \text{APRED}(S_i)$ " holds. Thus, two states are eliminated. These states are those associated with the predicates

$$S_{3x} = P_1x \& \neg P_2x \& P_3x$$

$$S_{4x} = P_1x \& \neg P_2x \& \neg P_3x$$

Therefore, there are a total of six possible student states in this example:

$$S_1x: P_1x \ \& \ P_2x \ \& \ P_3x$$

$$S_2x: P_1x \ \& \ P_2x \ \& \ \neg P_3x$$

$$S_5x: \neg P_1x \ \& \ P_2x \ \& \ P_3x$$

$$S_6x: \neg P_1x \ \& \ P_2x \ \& \ \neg P_3x$$

$$S_7x: \neg P_1x \ \& \ \neg P_2x \ \& \ P_3x$$

$$S_8x: \neg P_1x \ \& \ \neg P_2x \ \& \ \neg P_3x$$

Therefore, on the basis of Theorem 9, a unique subset of the set of all possible student states can be identified—namely, the set of possible student states. We will denote the set of possible student states by $\bar{S}$, where $\bar{S} \in S$.

The next question to be considered concerns the sequence of states a student can possibly go through during instruction. Since the student cannot enter an impossible state, only the set of possible student states, $\bar{S}$ need to be considered.

Within $\bar{S}$, we want to consider possible successors to student states, so $S_i \Rightarrow S_j$ is written to indicate that S_j is a possible successor to S_i . The restrictions to be placed upon the successor relation in the next axiom are two-fold. Consider the GCN-related attributes. The first restriction is that once a student has gained comprehension of a constituent, comprehension of that constituent is not lost. The emphasis here is on comprehension. It may be the case that the student has difficulty recalling some ideas associated with a constituent. However, it is natural to say in such a case that while the student comprehends the constituent, he or she needs to have his or her memory refreshed. Thus, it is reasonable to assert that comprehension of constituents is not lost over time.

The second restriction will be that constituent comprehension is gained one constituent at a time. It may be the case that the instructional process is arranged in such a way that comprehension of two or more constituents is gained simultaneously, *from observing the student*. However, the view maintained herein is that even though the student may *not* be *observed* gaining the constituents one at a time, the student nevertheless transverses the appropriate states in such a way that comprehension proceeds one constituent at a time. The difference here being what the instructional agent *observes* and the *actual* state of the student.

Neither of these restrictions implies that a student must gain constituent comprehension from instruction. That is, it is entirely possible for the student to fail to progress with respect to comprehension as a result of instruction. In this case, the student has not changed states for some reason. For example, the instruction might be inadequate.

Another possibility is that the student fails to synthesize previous knowledge in order to comprehend the constituent with which the instruction was concerned.

To summarize, instruction may or may not result in a state transition for a student. If a new state is entered, then the constituents comprehended in the old state must also be comprehended in the new state. Finally, the new state will assert that the student has gained comprehension of exactly one more constituent.

These notions are explicitly introduced in the next, and final axiom.

Axiom 12. $S_i \Rightarrow S_j$ if and only if a student in state S_j comprehends at least the same constituents as a student in state S_i , and comprehends, at most, one more constituent than he comprehends in state S_i .

Thus, for S_j to be a possible successor to state S_i , two conditions must be satisfied. First, a student in state S_j must comprehend the same constituents as he comprehended in state S_i . Second, the student must comprehend, in state S_j , at most, one more constituent than he comprehended in state S_i .

The next theorem gives three conditions for a state S_j to be a successor to another state S_i .

Theorem 10. For all $i, j = 1, \dots, 2^n$ if $S_i, S_j \in S$ then $S_i \Rightarrow S_j$ if and only if one of the following hold:

(i) $APRED(S_i) = APRED(S_j)$

(ii) $APRED(S_j) = APRED(S_i) \cup \{P_k\}$, where $P_k \notin APRED(S_i)$.

Theorem 10 provides the possible state transitions that we intuitively expect. First, the student might not enter a new state as a result of instruction, Part (i). Second, if the student enters a new state of comprehending new constituents, one new constituent can be comprehended and no constituent comprehension can be lost, Part (ii). Theorem 10 is concerned with the affirmative sets of GCN-related attributes. In each of part (i) and part (ii), a change may or may not occur in the non-GCN-related attributes possessed by a student. The nonsubject-matter attributes of a student may be vitally important from an instructional point of view. Thus, it is imperative that any model of state transitions include the possibility of transitions of this type.

The dependency relation within the GCN does not have to be explicitly evoked in order to determine possible successors. The dependency relation is used to restrict the number of states to be considered when possible successors are identified. In fact, only the set of possible states needs to be considered. That is, an impossible state cannot succeed any state, and thus impossible states are of no concern.

Notice also that nonsubject-matter-related attributes are of no concern when sequencing states. These attributes may or may not be attributable to the student, but

they do not restrict sequencing of states. As argued above, the purpose of instruction is not to modify these student attributes. The instructional process may take advantage of these attributes, and it may even affect them, but they do not restrict sequencing. For example, suppose constituent c_1 depends on constituent c_2 . Within the GCN, this is established independent of attributes, such as IQ. This means that whether a person's IQ is greater than 100 or not is independent of the states that will be gone through as a result of instruction when only subject-matter-related attributes are considered. In fact, no matter what the person's IQ, comprehension of c_2 will precede comprehension of c_1 , as established in the GCN dependency relation.

From an instructional point of view, an interesting question concerns how many and what paths there are through the subject matter. That is, how many independent paths are there that lead from comprehension of no constituents to comprehension of all constituents. Additionally, exactly what are these paths? In order to answer this question satisfactorily, we do not have to consider *all possible* paths based upon the possible successor relation. In fact, two considerations lead us to consider a proper subset of the set of all possible paths. The subset of possible paths results from eliminating two kinds of loops.

First, from the point of view of identifying possible paths through the subject matter we are not interested in counting stationary loops. That is, since a student can remain in the same state even after an instructional session, if such loops are counted, there is an obvious infinite number of possible transition paths. We want to be able to determine the paths through the subject matter independent of loops. Thus, we will only consider paths in which a state S_i cannot succeed itself. That is, while $S_i \Rightarrow S_i$ holds in general, such relations will not be included in potential paths through the subject matter.

Second, the concern is not with possible successors in which only the nonsubject-matter attributes are altered. In general, if we include successor relationships in which only nonsubject-matter attributes are altered, then a path containing such attribute alterations can, in general, result in infinitely long loops. One way to make this more clear is to consider the three predicate examples given above. Consider states

$$S_7x: \neg P_1x \ \& \ \neg P_2x \ \& \ P_3x$$

and

$$S_8x: \neg P_1x \ \& \ \neg P_2x \ \& \ \neg P_3x.$$

Then both $S_7 \Rightarrow S_8$ and $S_8 \Rightarrow S_7$ hold. Obviously, these transitions can lead to an infinite path. One possible infinite path is $S_7 \Rightarrow S_8 \Rightarrow S_7 \Rightarrow S_8 \Rightarrow \dots$. Such a path, while instructionally interesting, is not a path *through* the subject matter at all, and thus will be excluded.

Possible student paths will be further constrained in two ways not concerned with loops. First, it will be assumed that the student begins in a state in which no constituents are comprehended. This simply means that the student begins instruction being naive about the subject matter. Since all possible paths through the subject matter need to be identified, this naivete is a reasonable assumption to make. If a student is not naive concerning the subject matter, then the number of paths open to that student is reduced. The reduced set of possible paths in such a case is simply a proper subset of the totality of paths, and this subset can be identified readily.

Finally, it will be assumed that the student ends up in a state in which all constituents are comprehended. That is, the student terminal state is some state that is a member of G , the set describing the goal of instruction.

These considerations lead to the following definition.

Definition 14. A path through the subject matter is a list of states, $S_{i_1}, S_{i_2}, \dots, S_{i_k}$ such that,

$$(i) \text{ APRED}(S_{i_1}) = \emptyset,$$

$$(ii) \text{ For each } j = 2, 3, \dots, k, S_{i_{j-1}} \Rightarrow S_{i_j} \text{ and } S_{i_{j-1}} \neq S_{i_j}, \text{ and } \text{APRED}(S_{i_{j-1}}) \neq \text{APRED}(S_{i_j}),$$

$$(iii) S_{i_k} \in G.$$

In summary, part (i) asserts the beginning state of a path to be the state in which no subject-matter constituents are comprehended, Part (ii) asserts that for two adjoining different states in the path, the latter state must be a successor to the former state, and the affirmative predicate sets of the two states must be different, part (iii) asserts the final state of the path to be one of the possible terminal states (the goal).

One way to answer the question concerning how many paths there are through the subject matter is to develop a graph containing all the possible paths. This can be done through the following procedure. The procedure results in a multilevel graph in which successive levels contain possible successors as constrained by part (ii) of Definition 8.

Procedure.

Step 1. For level 1, include every state S_i such that

$$\text{APRED}(S_i) = \emptyset.$$

Step 2. For each level k ($k \geq 2$), include in level k every possible successor S_j to state S_i , where S_i is included in level $k-1$, $S_i \neq S_j$ and $\text{APRED}(S_i) \neq \text{APRED}(S_j)$.

Graphically, this process can be done by first writing all states in which no constituents are comprehended in a horizontal line (Step 1). On the line below this, for

each state S_i on the first line, write (horizontally) all states S_j such that $S_i \Rightarrow S_j$ obtains, $S_i \neq S_j$ and $\text{APRED}(S_i) \neq \text{APRED}(S_j)$. Also when $S_i \Rightarrow S_j$ obtains, connect S_i to S_j by a directed line. In the same way construct further lines. It can be shown that this process will terminate at some line, p , where every entry, S_i , in line p is such that $S_i \in G$, or $\text{APRED}(S_i) = \{P_1, \dots, P_m\}$. That is, every state on line p asserts that an individual in that state comprehends all GCN constituents.

Three examples are presented next to illustrate this important procedure. All the examples will be based on three predicates, P_1, P_2, P_3 , but the examples will differ in the way the corresponding three constituents, c_1, c_2, c_3 , depend upon each other and which of the constituents are GCN- or non-GCN-related. The first example will be discussed in greater detail than will the second and third examples. For reading ease, the eight S predicates, corresponding to the three predicates, P_1, P_2, P_3 , are repeated below:

- $S_1x: P_1x \ \& \ P_2x \ \& \ P_3x$
- $S_2x: P_1x \ \& \ P_2x \ \& \ \neg P_3x$
- $S_3x: P_1x \ \& \ \neg P_2x \ \& \ P_3x$
- $S_4x: P_1x \ \& \ \neg P_2x \ \& \ \neg P_3x$
- $S_5x: \neg P_1x \ \& \ P_2x \ \& \ P_3x$
- $S_6x: \neg P_1x \ \& \ P_2x \ \& \ \neg P_3x$
- $S_7x: \neg P_1x \ \& \ \neg P_2x \ \& \ P_3x$
- $S_8x: \neg P_1x \ \& \ \neg P_2x \ \& \ \neg P_3x$.

Example 1

Suppose c_1 and c_2 are GCN constituents, and c_3 is a non-GCN-constituent; in addition, suppose c_1 depends on c_2 , as shown in Figure 19. The set of possible states, $\bar{S}$,

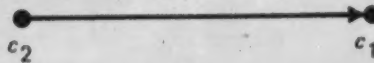


Figure 19. GCN where c_1 depends on c_2 .

has six members: $S_1, S_2, S_5, S_6, S_7, S_8$; the impossible states being S_3 and S_4 . Step 1 of the procedure gives the situation shown in Figure 20 for line 1. The first application of Step 2 extends the graph to include line 2 to the one shown in Figure 21. A second



Figure 20. First step of the procedure.

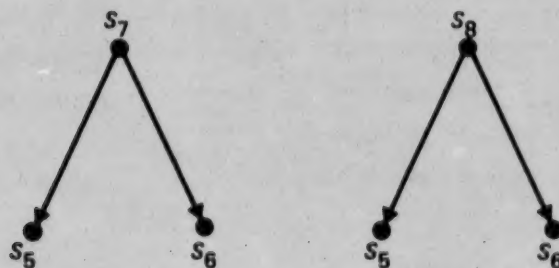


Figure 21. Iteration 1 of second step of procedure.

application of Step 2 yields the graph shown in Figure 22. Notice that every state S_i in the bottom line of Figure 22 is also that $S_i \in G$. Thus, there are no successors to any state in level 3 to be considered. That is, in this example, only two applications of Step 2 of the procedure are required since level 3 contains only states that do not have successors.

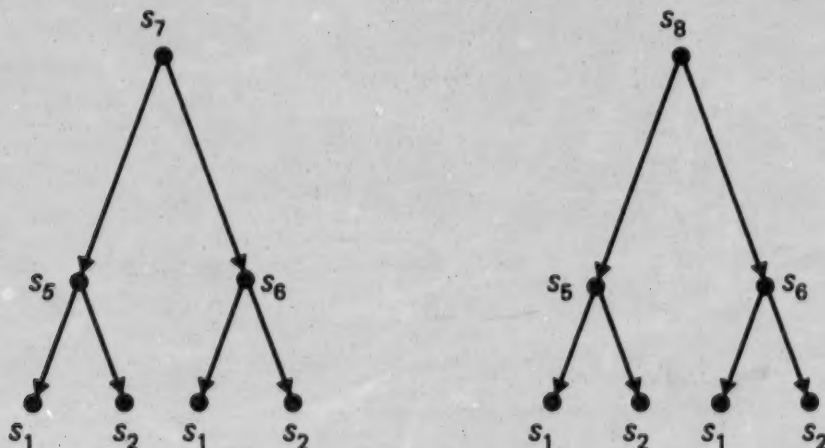


Figure 22. Iteration 2 of second step of procedure:
the possible paths through the GCN of Figure 19.

In order to determine the total number of paths through the subject matter, it is only necessary, at this point, to count the unique paths leading from level 1 to a state S_i where $S_i \in G$. In this case, there are a total of 8 paths through the subject matter.

This simple example serves to bring out one final point concerning the definition of the notion of a path through the subject matter. Notice that the two sequences S_7, S_5, S_1 and S_7, S_6, S_1 are identical when only subject-matter attributes are considered. In both cases, a student goes from comprehension of no constituents to comprehension of c_2 only, and finally to comprehension of both c_1 and c_2 . At issue here is whether or not it is necessary to distinguish states S_5 and S_6 . From the point of view of progression

through the subject matter itself, the answer is no. Definition 8 (and the graph procedure) could be modified so that S_5 and S_6 are not distinguished. In this case, the number of possible paths is reduced from 8 to 4. Pursuing this reasoning, there would be no need to distinguish S_7 and S_8 , and no need to distinguish S_1 and S_2 . Thus, we are left with only one path, in essence.

Clearly, this is not intuitively correct. The problem is that while the paths through the subject matter are identified in a strict sense, relevant instructional information is lost. We need to identify what state the student is in when instruction commences: S_7 or S_8 . The nonsubject-matter attributes are instructionally important at this stage. When the student gains comprehension of c_2 , we need to know what state is entered: S_5 or S_6 . Again the nonsubject-matter attributes are important concerning subsequent instruction.

Therefore, Definition 8 and the procedure based on it provide us with precisely the information concerning the paths through the subject matter we expect. The procedure identifies possible paths that consist of possible successive states. The states that are identified in the paths, in turn, provide all student-attribute data relevant for instructional purposes. Thus, an effective procedure shows how many and what paths there are through the subject matter.

Example 2

Suppose all three constituents are GCN constituents and that both c_2 and c_3 depend upon c_1 as indicated in Figure 23. In this case, S_5 , S_6 , and S_7 are all impossible student

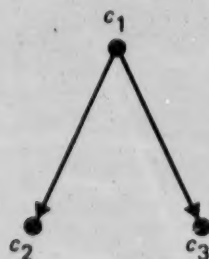


Figure 23. GCN where both c_2 and c_3 depend on c_1 .

states. Applying the procedure yields the graph shown in Figure 24. From the figure, the number of paths through the subject matter is 2.

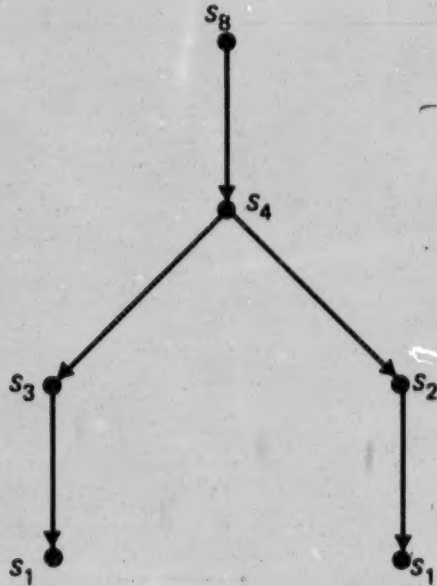


Figure 24. The possible paths through the GCN of Figure 23.

Example 3

Suppose the three constituents are all GCN-related and that c_3 depends upon both c_1 and c_2 as shown in Figure 25. For this situation, s_3 , s_5 and s_7 are all impossible student states. Applying the procedure gives the path diagram shown in Figure 26 with two paths through the subject matter.

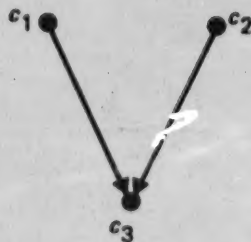


Figure 25. GCN where c_3 depends on both c_1 and c_2 .

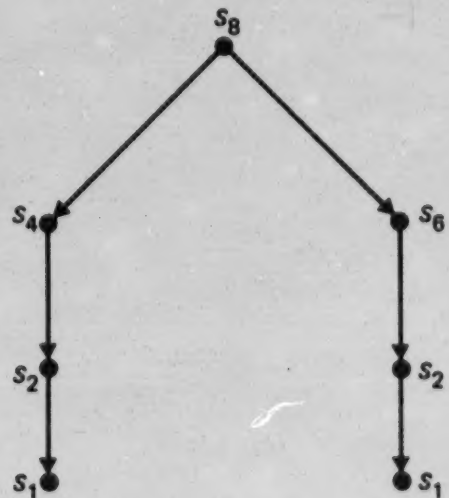


Figure 26. The possible paths through the GCN of Figure 25.

Chapter 4

AN APPLICATION

A concrete example of the foregoing theoretical developments is given in this chapter. The example concerns the application of the theoretical model to the Apprentice Still Photographic Camera Specialist Course, hereafter referred to as the "photo course." The photo course consists of a five-volume, nonindividualized set of instruction manuals¹—23631 01 1171, 23631 02 1071, 23631 03 1271, 23631 04 7201, and 23631 05 7203.

Referring to Figure 1, the general subject matter consists of photographic principles concerning the use of a still camera, film, and processing and printing. The context of instruction includes all Air Force personnel eligible for such a course. In addition, the course assumes a *slight* understanding of basic chemistry and basic quantitative mathematics. The mathematics required include arithmetic and elementary algebra.

A few preliminary comments are in order before turning to the GCN and TICS. It is not our purpose to develop a subject-matter structure along the lines of the GCN/TICS model for a photo course. This certainly could be done. The problem being addressed is to demonstrate that the *given* subject matter for the photo course can be recast into the GCN/TICS model as described earlier. That is, we must show that the course material as *given* can be recast in terms of the described theoretical entities (constituents, etc.).

For the purpose of this report, it will not be necessary to deal with the complete photo course. It will suffice to consider only a portion of the course. However, a sufficiently large segment of the course will be covered. This will be done so that there will be no doubt that the process could, in theory, be extended to cover the complete photo course.

In the discussion that follows, references are made to the given course material by volume and page as appropriate. For example, one Basic Principle concerns the refraction of oblique light when it passes from one medium to a medium of different density. This principle is stated in Volume II, p. 38. References of this type will be noted by (II,38). These references will aid the interested reader in locating the constituents that are identified in the course material.

¹ Available from The Air University, Gunter Air Force Base, Montgomery, Alabama.

The photo course consists of many Primary and Secondary Notions, and Basic and Established Principles. The material has many interconnections and relationships between constituents. As written, it is precise and extremely detailed. It is stressed that the following description is not intended to be a course in itself. It is an attempt to show that the photo course can fit the axiomatic subject-matter structure model under consideration.

Only a portion of the photo course has been analyzed. For example, the concepts associated with emulsions have been traced to the most primitive constituents. The concept of lenses are discussed in great detail in the course (Volume II). We have not developed the complete structure underlying lenses, but we have indicated briefly the Notions and Principles underlying this Secondary Notion.

As written, the photo course is nonlinear in that concepts are introduced, briefly explained, and later explained in detail. Thus, one notion may be introduced several times, with each explanation providing a deeper understanding of the notion. From a subject-matter structuring point of view, repetition must be eliminated. For instruction purposes, repetition is important and often highly desirable. Since we are interested in subject-matter structure, it follows that we must integrate the material by first eliminating repetition. This will be the first task, after which the formal structure will be developed.

Elementary Photography is conceived as being a three-step process (I,20), Exposure, Processing, and Printing. Exposure consists of light rays (II,15), reflected (I,20) from a subject (I,20), passing through the lens (I,21), diaphragm (I,21), and shutter (I,21) to form a latent image (I,20) in the film (I,20). The film is an emulsion (I,20) consisting of silver halide grains (II,4) in a gelatin (I,20). The camera (I,20) is the device containing the lens, diaphragm, shutter, and film. Silver halide grains in the film have varying sensitivity to light (II,4). Emulsion speed (II,3) is a measure of how fast the emulsion material reacts to light. Emulsion film speeds are measured numerically so that higher numbered emulsions are more sensitive (II,4). Silver halide grains in the film also tend to clump together (II,4). This clumping determines the size of the grain in the film (II,4). Higher speed films tend to have larger grains (II,4). Inherent grain size (II,4) is the tendency of a particular film to produce a grain of a certain size. Granular image structure (II,4) or grain size also affects the sharpness of the image. Small regular emulsion grains yield sharper images (II,4). Resolving power (II,4) is the inherent property of an emulsion to reproduce fine detail. Resolving power is numerically measured by expressing the number of lines per millimeter that can be individually distinguished in the photographic image (II,4). The

smaller the distance between individual lines, the greater the number of lines per millimeter, the greater resolving power, and the sharper the image detail (II,4).

Darkness is formed in the film emulsion as a result of exposure (II,5). The darkness, or light-stopping ability, is referred to as density (II,5). Contrast is the difference between high- and low-density areas. Bright subject areas cause heavy density, while dark subject areas cause light density (II,5).

Processing involves the use of developer (I,20) to create visible black metallic silver (I,20), from the exposed emulsion. The film goes through a rinse bath (I,20), a hypo solution (I,20) or fixer (I,24), and is washed (I,20). As a result, the latent image becomes a visible negative image (I,20). Drying is the last processing step (I,20).

Printing involves passing transmitted light (I,25) through the negative onto a paper emulsion (I,20). Print processing involves development (I,23), fixing (I,24), washing (I,25), and drying (I,25).

The Processing and Printing notions can be traced down further, but, as mentioned above, it will not be done. This brief sketch is concluded with two principles concerning contrast (II,7), and two principles concerning laboratory light (I,20).

- (1) High contrast subjects produce high contrast negative images regardless of the type of film being used.
- (2) Negative image contrast varies directly with development time.
- (3) Laboratory light must be controlled during processing.
- (4) Laboratory light must be controlled during printing.

On the basis of this summary of the photo course, ten Primary Notions can be identified.

- PN1: light rays
- PN2: reflection
- PN3: subject
- PN4: silver halide grains
- PN5: gelatin
- PN6: sensitivity to light
- PN7: inherent grain size
- PN8: granular image structure
- PN9: visible black metallic silver
- PN10: transmitted light

On the basis of these 10 Primary Notions (and others not given in the above summary) 30 Secondary Notions are developed.

- SN1: lens—a device for collecting and focusing light rays reflected from the subject.
- SN2: diaphragm—a device controlling the amount of light that passes through the lens.
- SN3: emulsion—light-sensitive portion of photographic material.
- SN4: emulsion grain—the microscopic light-sensitive particles in an emulsion.
- SN5: film—emulsion consisting of silver halide grains in a gelatin.
- SN6: shutter—a device that controls the length of time it takes for light from the lens to strike the film.
- SN7: latent image—the image of the subject formed in an emulsion.
- SN8: camera—a device consisting of lens, diaphragm, shutter, and film holder.
- SN9: emulsion speed—a measure of how fast the emulsion reacts to light.
- SN10: resolving power—the inherent property of the emulsion to reproduce fine detail; measured numerically; depends on inherent grain size and granular image structure.
- SN11: film exposure—the process of forming the latent image in the film from light rays reflected from the subject into the camera.
- SN12: density—darkness in the film emulsion resulting from exposure.
- SN13: high density—dark areas.
- SN14: low density—lighter areas.
- SN15: contrast—difference between high- and low-density areas.
- SN16: film development—chemical process used to create a visible black metallic silver image from the exposed film emulsion.
- SN17: film rinse bath—used to wash developer from film after film development.
- SN18: film hypo solution or fixer—chemical used to dissolve remaining film emulsion after development.
- SN19: film wash—washing chemicals from film after fixing.
- SN20: visible negative image—film image that results from development, rinsing, fixing, and washing exposed film.
- SN21: film drying—the last step in developing film.
- SN22: processing—the process of forming a visible negative image from the exposed film by developing, rinsing, fixing, washing, and drying.

- SN23: paper emulsion—an emulsion contained in paper.
- SN24: print exposure—transmitting light through the visible negative image in order to create a latent image in the print emulsion.
- SN25: print development—chemical process used to create a visible image in the exposed print.
- SN26: print fixing—dissolving unused emulsion in the paper.
- SN27: print washing—washing the visible image print after fixing.
- SN28: print drying—drying the print after washing.
- SN29: printing—the process consisting of print exposure, print development, print fixing, print washing, and print drying.
- SN30: print—the final product that results from exposure, processing, and printing.

The seven Basic Principles that are used to relate these notions consist of the following:

- BP1: silver halide grains in film emulsion have varying sensitivity to light that can be numerically measured.
- BP2: silver halide grains in film tend to clump together determining grain size in the film.
- BP3: small regular emulsion grains yield sharper images.
- BP4: darkness is formed in the film emulsion as a result of exposure.
- BP5: bright subject areas cause heavy density, while light subject areas cause low density.
- BP6: darkness is formed in the paper emulsion as a result of print exposure.
- BP7: high-density film areas result in less exposed paper emulsion, while lower-density film areas cause more exposure in the paper emulsion.

Finally, there are 12 Established Principles:

- EP1: higher numbered film emulsions have greater light sensitivity.
- EP2: higher speed film emulsion tends to have larger grains.
- EP3: the greater the resolving power the sharper the image detail.
- EP4: light subject areas are dark in the film, but dark subject areas are light in the film.
- EP5: since printing is on white paper, dark (high density) film areas remain white, but light (low density film areas) become dark.
- EP6: light subject areas are light in the print and dark subject areas are dark in the print.

EP7: Thus, the print is an image of the subject.

EP8: light that gives brilliant subject highlights and dark black subject shadows produces higher-contrast visible negative images.

EP9: high-contrast subjects produce higher-contrast negative images, regardless of the type of film being used.

EP10: negative-image contrast varies directly with development time.

EP11: since film emulsion is light-sensitive before processing, laboratory light must be controlled during processing.

EP12: since paper emulsion is light-sensitive before print fixing, laboratory light must be controlled during printing.

Not every notion used in the Secondary Notions, Basic Principles, and Established Principles has been sorted out and included in the list of Primary and Secondary Notions. For example, EP11 and EP12 make use of the concept of controlling light, which has not been introduced or explained. This could be done, but for the purposes of this report, this effort does not seem justified. The listing of the 59 constituents given for the photo course should be inclusive enough to demonstrate how to begin to apply the theoretical model described earlier.

The next task is to describe the relations used to interrelate these 59 constituents. There are two binary relations defined on the set of constituents: (a) "x is used to formulate y," and (b) "x is used to establish y". First, consider the "used in the formulation of" relation. In this relation, x can be either a Primary or Secondary Notion, while y can be a Secondary Notion, a Basic Principle, or an Established Principle. A tabular summary of the "used in the formulation of relation" is given and every Secondary Notion, Basic Principle, and Established Principle is listed in Table 14. The list of constituents in the row for any of these entries is a list of all constituents that are used in the formulation of that constituent. For example, consider SN30, the concept of a print. From the list, one sees that:

- (1) SN11 is used in the formulation of SN30.
- (2) SN22 is used in the formulation of SN30.
- (3) SN29 is used in the formulation of SN30.

Now consider the "used to establish" relationship: "x is used to establish y". In this case, x can be any constituent, a Primary Notion, Secondary Notion, Basic Principle, or Established Principle. On the other hand, the only possible constituents that can take the place of y are Established Principles. Table 15 presents a tabular summary of the "used to

Table 14. Tabular Representation of the
"Used in the Formulation of" Relation

SN1	PN1, PN2, PN3
SN2	PN1, SN1
SN3	PN6
SN4	PN6, SN3
SN5	PN4, PN5, SN3
SN6	PN1, SN1, SN5
SN7	PN3, SN3
SN8	SN1, SN2, SN6
SN9	PN1, SN3
SN10	PN7, PN8, SN3
SN11	PN1, PN2, PN3, SN5, SN7, SN8
SN12	SN3, SN5, SN10
SN13	SN12
SN14	SN12
SN15	SN13, SN14
SN16	PN9, SN3, SN5, SN11
SN17	SN5, SN16
SN18	SN3, SN16
SN19	SN5, SN18
SN20	SN5, SN16, SN17, SN18, SN19
SN21	SN5
SN22	SN5, SN11, SN16, SN17, SN18, SN19, SN20, SN21
SN23	SN3
SN24	PN10, SN7, SN20, SN23
SN25	SN24
SN26	SN3, SN23
SN27	SN26
SN28	SN27
SN29	SN24, SN25, SN26, SN27, SN28
SN30	SN11, SN23, SN29
BP1	PN4, PN6, SN5
BP2	PN4, SN5
BP3	PN7, PN8, SN4
BP4	SN3, SN5, SN11
BP5	PN3, SN13, SN14
BP6	SN23, SN24
BP7	SN5, SN13, SN14, SN23, SN24
EP1	PN6, SN3, SN5
EP2	SN3, SN4, SN5, SN9
EP3	SN10
EP4	PN3, SN5
EP5	SN5, SN13, SN14, SN29
EP6	PN3, SN30
EP7	PN3, SN30
EP8	PN1, PN3, SN15, SN20
EP9	SN5, SN15, SN20
EP10	SN15, SN16, SN20
EP11	PN1, PN6, SN3, SN5, SN22
EP12	PN1, PN6, SN23, SN26, SN29

Table 15. Tabular Representation of the
"Used to Establish" Relation

EP1	PN6, SN3, SN5, BP1
EP2	SN3, SN4, SN5, SN9, BP1, BP2, EP1
EP3	PN7, PN8, SN10, BP3
EP4	PN3, SN5, BP4, BP5
EP5	SN5, SN13, SN14, SN29, BP6, BP7
EP6	PN3, SN30, EP4, EP5
EP7	PN3, SN30, EP6
EP8	PN1, PN3, SN15, SN20, EP4
EP9	PN3, SN5, SN15, SN20, EP8
EP10	SN15, SN16, SN20
EP11	PN1, PN6, SN3, SN5, SN22, EP4
EP12	PN1, PN6, SN23, SN26, SN29, EP5

establish" relation. The interpretation of Table 15 is similar to the interpretation of Table 14. For example, Secondary Notion SN10 and Basic Principle BP3 are used to establish Established Principle EP3, as well as the Primary Notions PN7 and PN8.

Because a complete analysis of the photo course has not been provided, Table 15 is somewhat incomplete. That is, it may not be possible to establish completely a given Principle on the bases of the constituents listed. This follows, since the argument given in the photo course to establish a Principle may depend upon some constituents that have not been singled out for attention herein. If the analysis of the photo course was performed completely, then this shortcoming would be eliminated. For the purposes of this report, this is unessential.

Figures 27 and 28 provide a graphic representation of the data presented in Tables 14 and 15, respectively. Thus, Figure 27 provides a graphic representation of the "used in the formulation of" relation. A directed line leading from one box in Figure 27 to another indicates that the constituent represented in the first box is used in the formulation of the constituent represented in the second box. In Figure 27 (and in Figure 28) a dot on the intersection of two lines indicates that they are connected. The lack of a dot indicates the lines merely cross and are not connected.

To interpret Figure 27, consider for example PN1. Following the line from the bottom of the PN1 box leads to a two-way connection. Following the right branch, PN1 is used in the formulation of EP8, EP11, and EP12. Returning to the junction and following the branch to the bottom of the figure, it shows that PN1 is used in the formulation of SN1, SN2, SN6, SN9, and SN11. Figure 27 can also be used to determine which constituents are used in the formulation of a given constituent. Consider, for example, EP11. The arrow to the left of the EP11 box leads back to PN1 and PN6. The

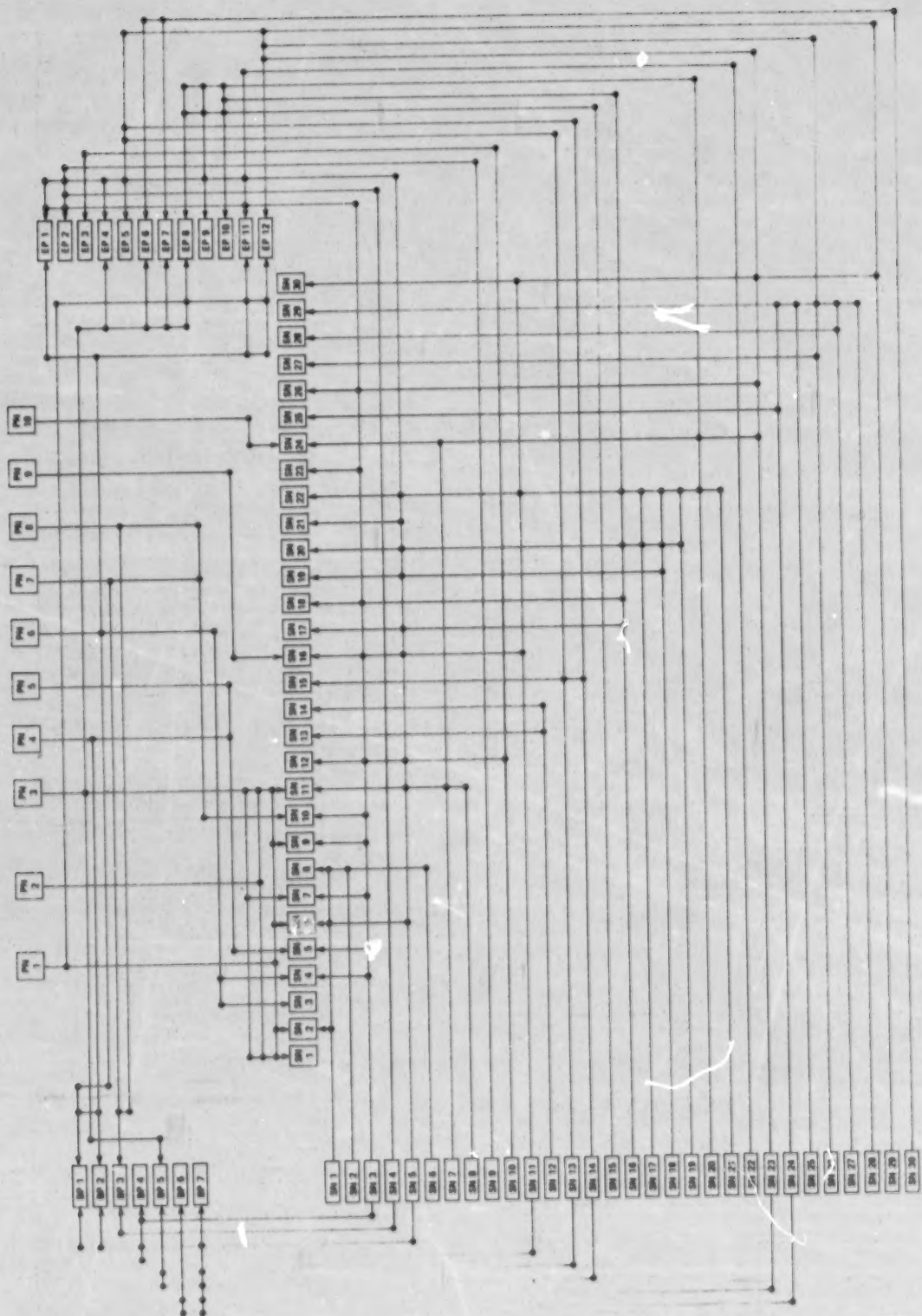
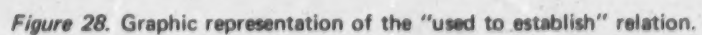


Figure 27. Graphic representation of the "used in the formulation of" relation.



arrow to the right of the EP11 box leads to SN3, SN5, and SN22. That these are the only constituents used to formulate EP11 can be verified by the entries in Table 14.

The interpretation of Figure 28 is entirely similar to the interpretation of Figure 27. For example, consider the following question: "Secondary Notion 20 (SN20) is used in the establishment of which Established Principles (EP)?" Figure 28 shows that SN20 is used in the establishment of EP8, EP9, and EP10.

The Adjacency Matrices for the two GCN relationships are given in Figures 29 and 30. Figure 29 is the Adjacency Matrix for the "used in the formulation of" relation, and Figure 30 is the Adjacency Matrix for the "used to establish" relation. As explained in Chapter 1, if an entry of 1 appears in an Adjacency Matrix, then the constituent in that row is adjacent to the constituent in the column. In Figure 29, this means the row constituent is used in the formulation of the column constituent. In Figure 30 the row constituent is used to establish the column constituent.

Figures 29 and 30 are modified to make them square, such that rows and column heads are identical, and they are presented in the computer printout Figures 31 and 32.

The matrices in Figures 31 and 32 are combined by Boolean arithmetic to produce the adjacency matrix $A(D)$ for the dependency relation in the photo course GCN as shown in Figure 33.

Consider Secondary Notion 1 in Figure 33. Reading down the SN1 column we see that SN1 is dependent on PN1, PN2, and PN3. Furthermore, this dependency is such that for each of the listed constituents there is a GCN relation leading from the constituent to SN1. From Figure 33 it is not possible to determine *which* GCN relation serves to relate these constituents to SN1. This information is contained in Figures 31 and 32.

In the case of the photo course, it was found that $A^{16}(D) = [0]$. Thus, the maximal length dependency chain for the portion of the photo course that was analyzed is 15. That is, $m = 15$, hence the longest possible sequence of dependency relations contains 15 arcs or links. For completeness we have displayed $A^2(D)$, $A^3(D)$, ..., $A^{15}(D)$ in Figures 34 through 47, respectively. Figure 48 displays the Boolean sum of $A(D)$, $A^2(D)$, ..., $A^{15}(D)$. That is, Figure 48 is the matrix $A^*(D)$, where

$$A^*(D) = \left[\sum_{i=1,15} A^i(D) \right] \#$$

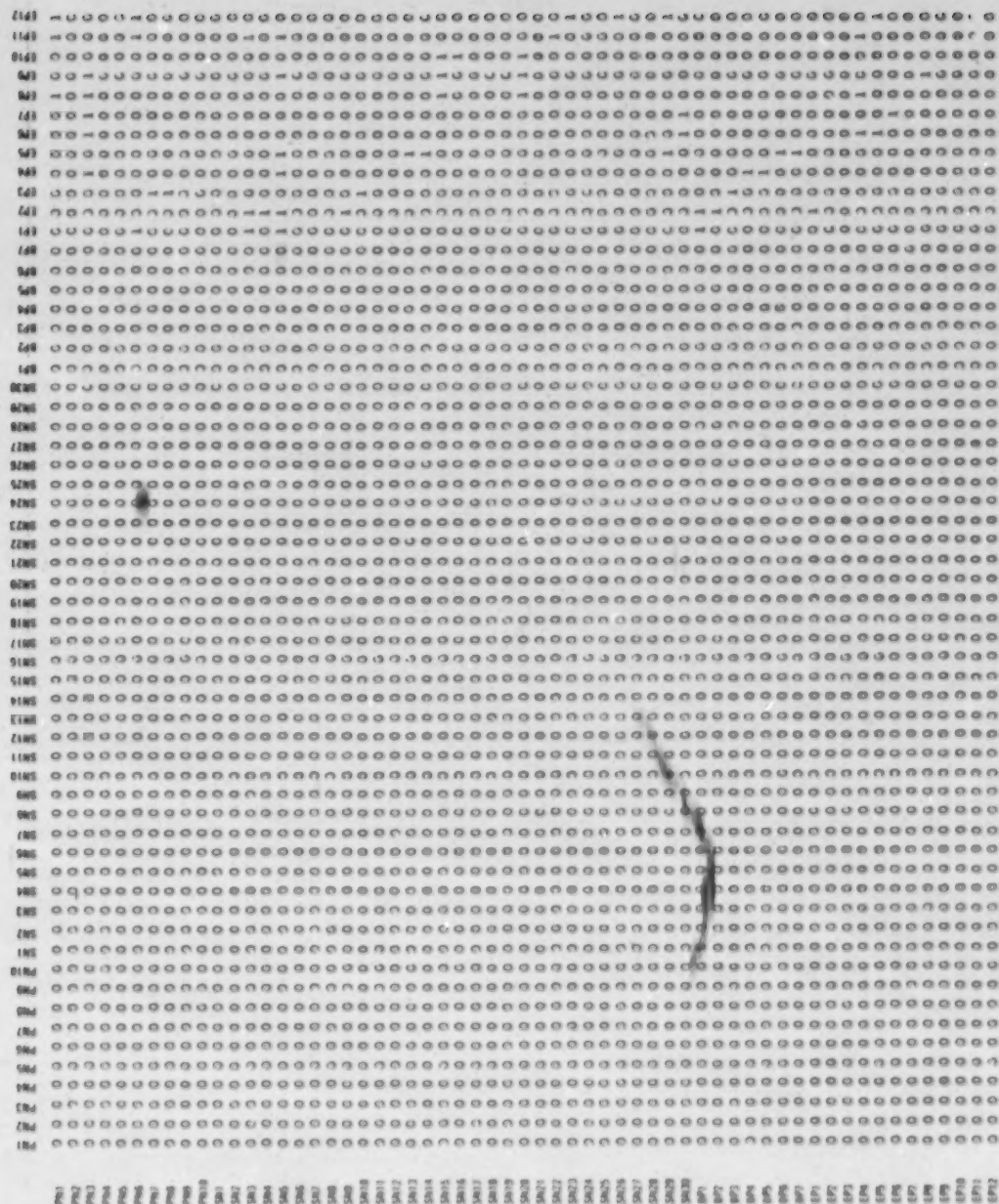
Figure 48 then presents a complete summary of the GCN dependency relation. For example, as seen from Figure 48, EP12 depends on PN1, PN2, PN3, PN4, PN5, PN6, PN7, PN8, PN9, PN10, SN1, SN2, SN3, SN5, SN6, SN7, SN8, SN10, SN11, SN12, SN13, SN14, SN16, SN17, SN18, SN19, SN20, SN23, SN24, SN25, SN26, SN27, SN28, SN29, BP6, BP7, and EP5.

	EP 1	EP 2	EP 3	EP 4	EP 5	EP 6	EP 7	EP 8	EP 9	EP 10	EP 11	EP 12
PN 1								1			1	1
PN 2												
PN 3				1		1	1	1	1			
PN 4												
PN 5												
PN 6	1										1	1
PN 7			1									
PN 8			1									
PN 9												
PN 10												
SN 1												
SN 2												
SN 3	1	1									1	
SN 4		1										
SN 5	1	1		1	1				1		1	
SN 6												
SN 7												
SN 8												
SN 9		1										
SN 10			1									
SN 11												
SN 12												
SN 13					1							
SN 14					1							
SN 15								1	1	1		
SN 16										1		
SN 17												
SN 18												
SN 19												
SN 20								1	1	1		
SN 21												
SN 22											1	
SN 23												1
SN 24												
SN 25												
SN 26												1
SN 27												
SN 28												
SN 29					1							1
SN 30						1	1					
BP 1	1	1										
BP 2		1										
BP 3			1									
BP 4				1								
BP 5				1								
BP 6					1							
BP 7					1							
EP 1		1										
EP 2												
EP 3												
EP 4						1		1			1	
EP 5						1						1
EP 6							1					
EP 7												
EP 8									1			
EP 9												
EP 10												
EP 11												
EP 12												

Figure 30. Adjacency matrix for the "used to establish" relation

[illegible]

Figure 31. Adjacency matrix for the "used in the formulation of" relation R .



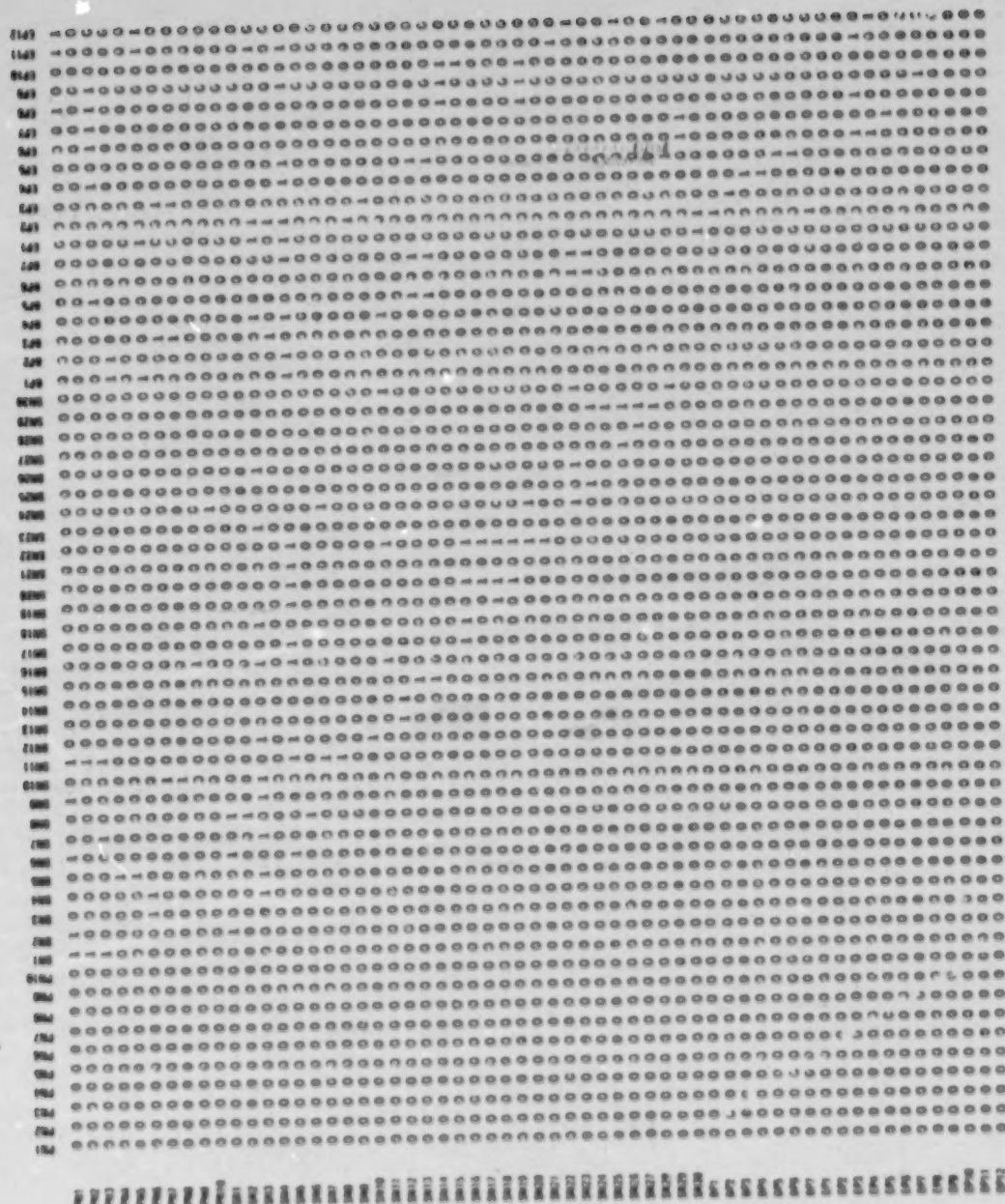


Figure 33. Adjacency for the dependency relation in the photo course GCN, A/D.

[illegible]

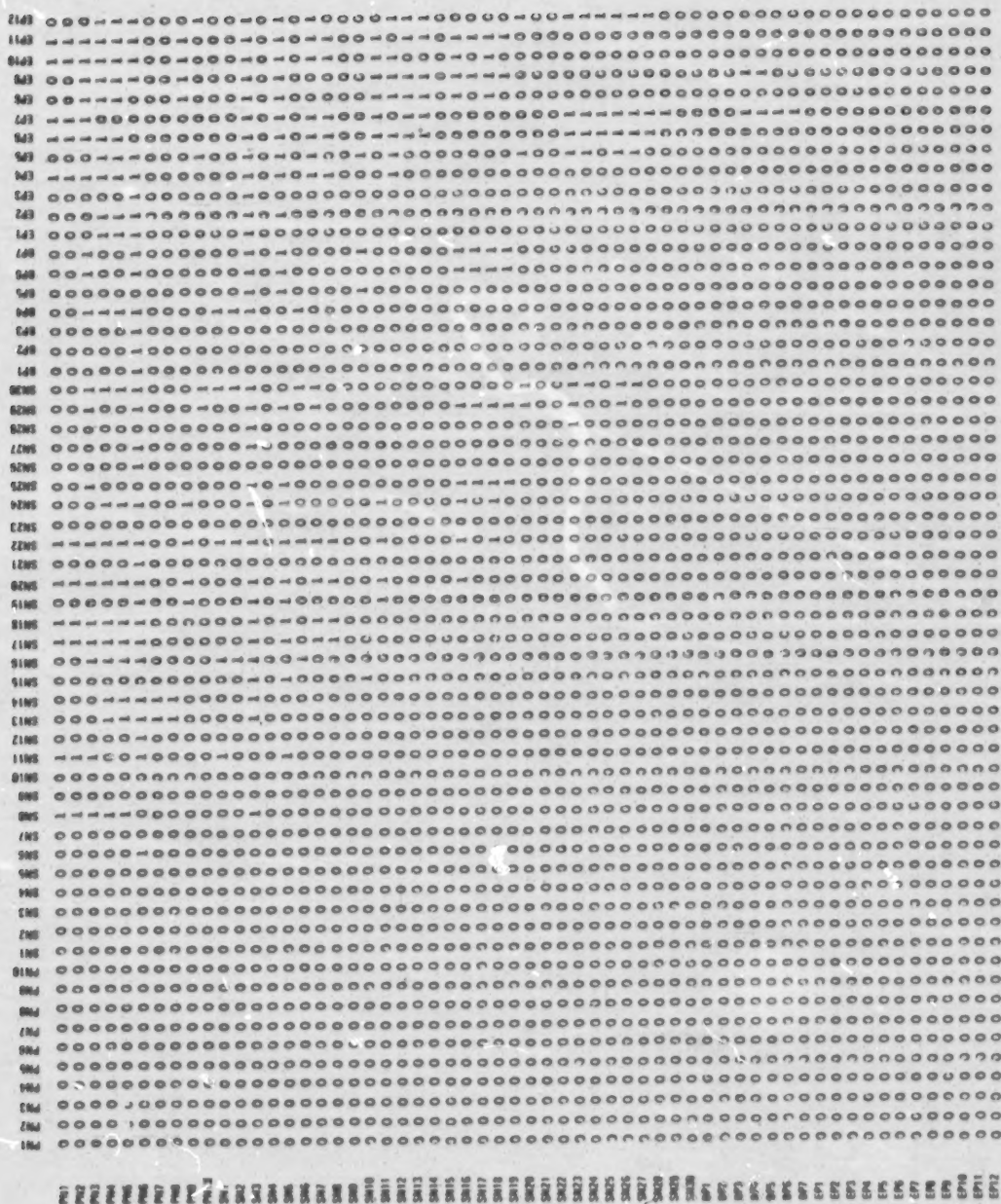


Figure 35. A³(D).

Figure 36. $A^4(D)$.

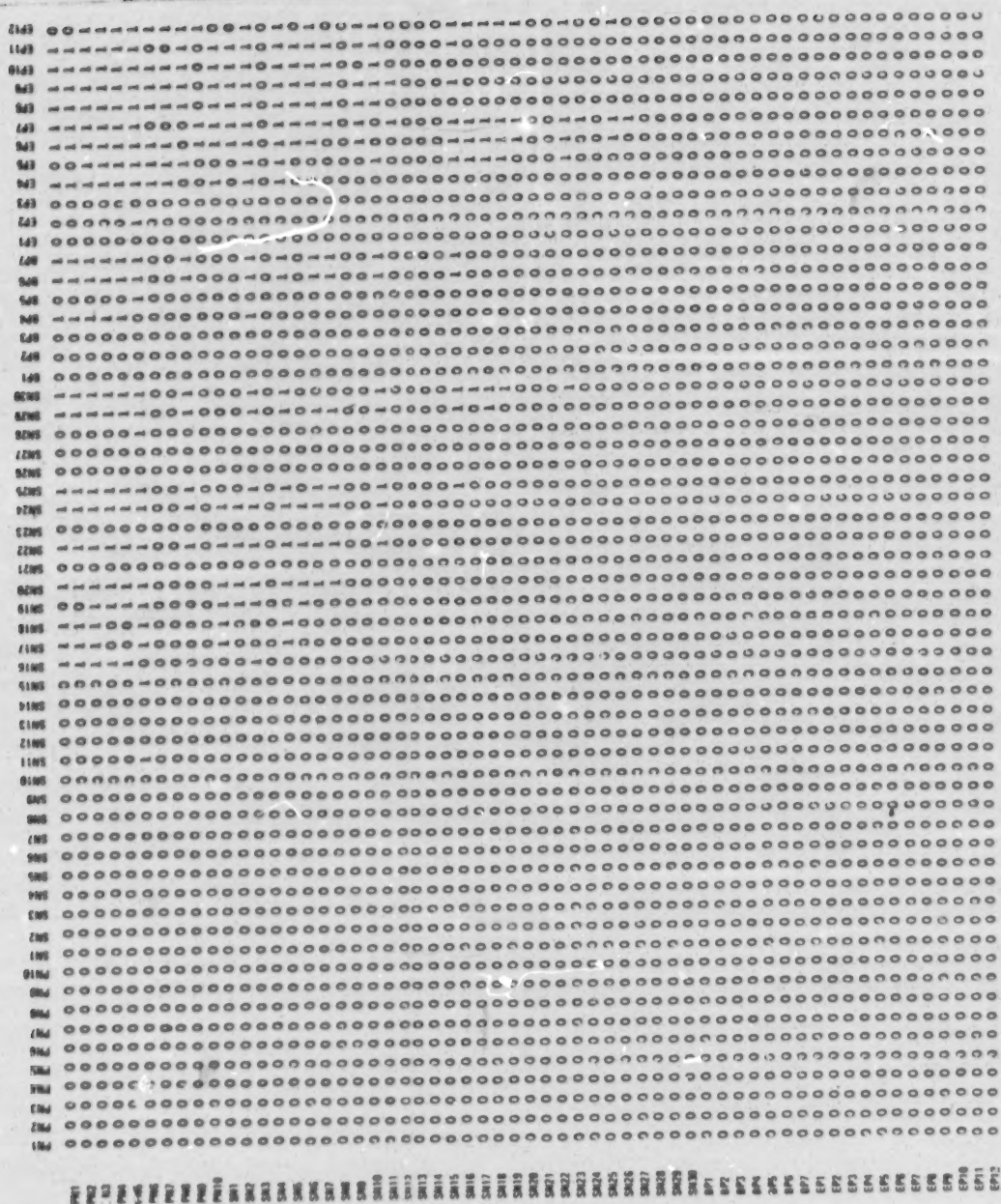


Figure 38. $A^6(D)$.

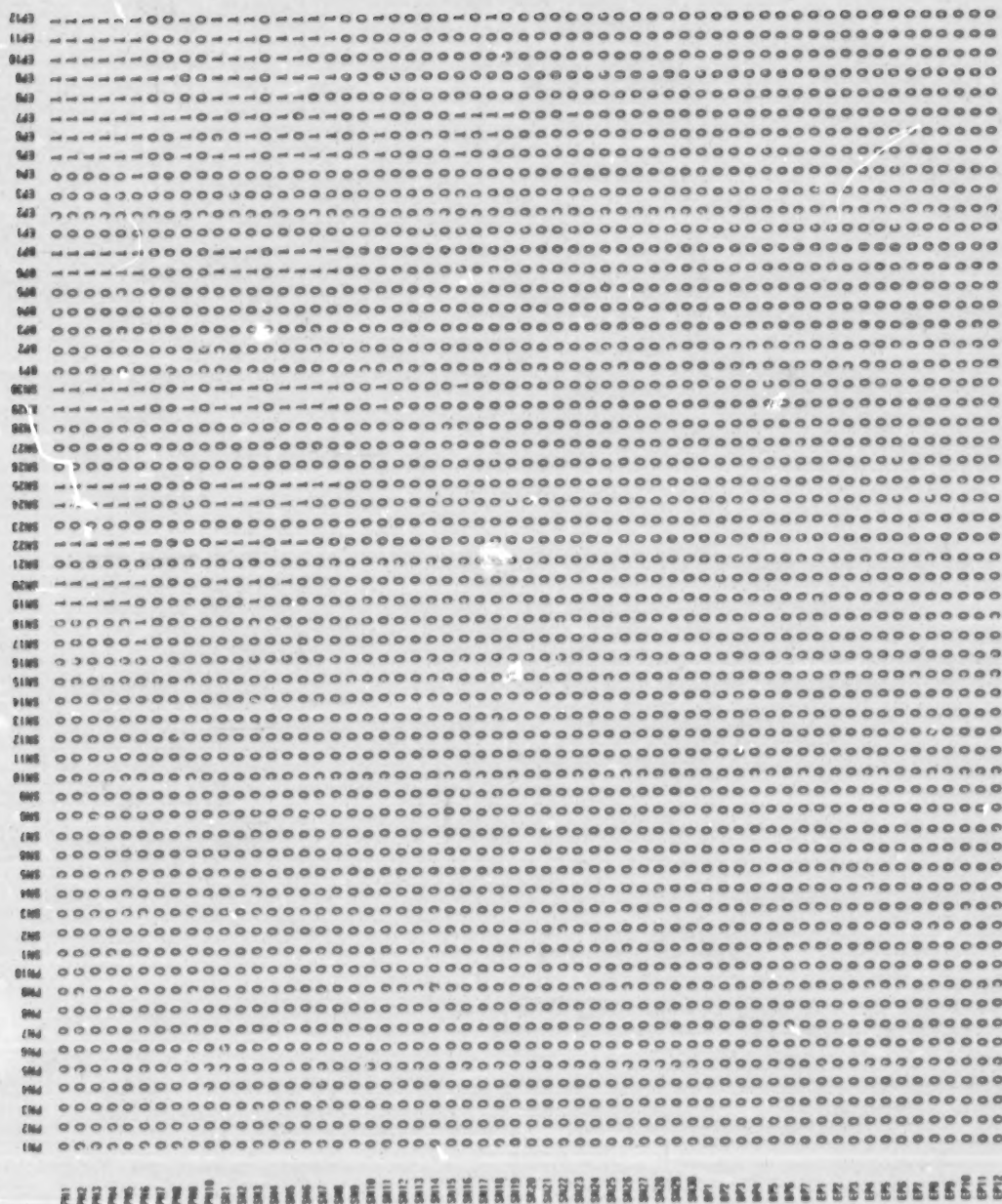


Figure 39. $A^7(D)$.

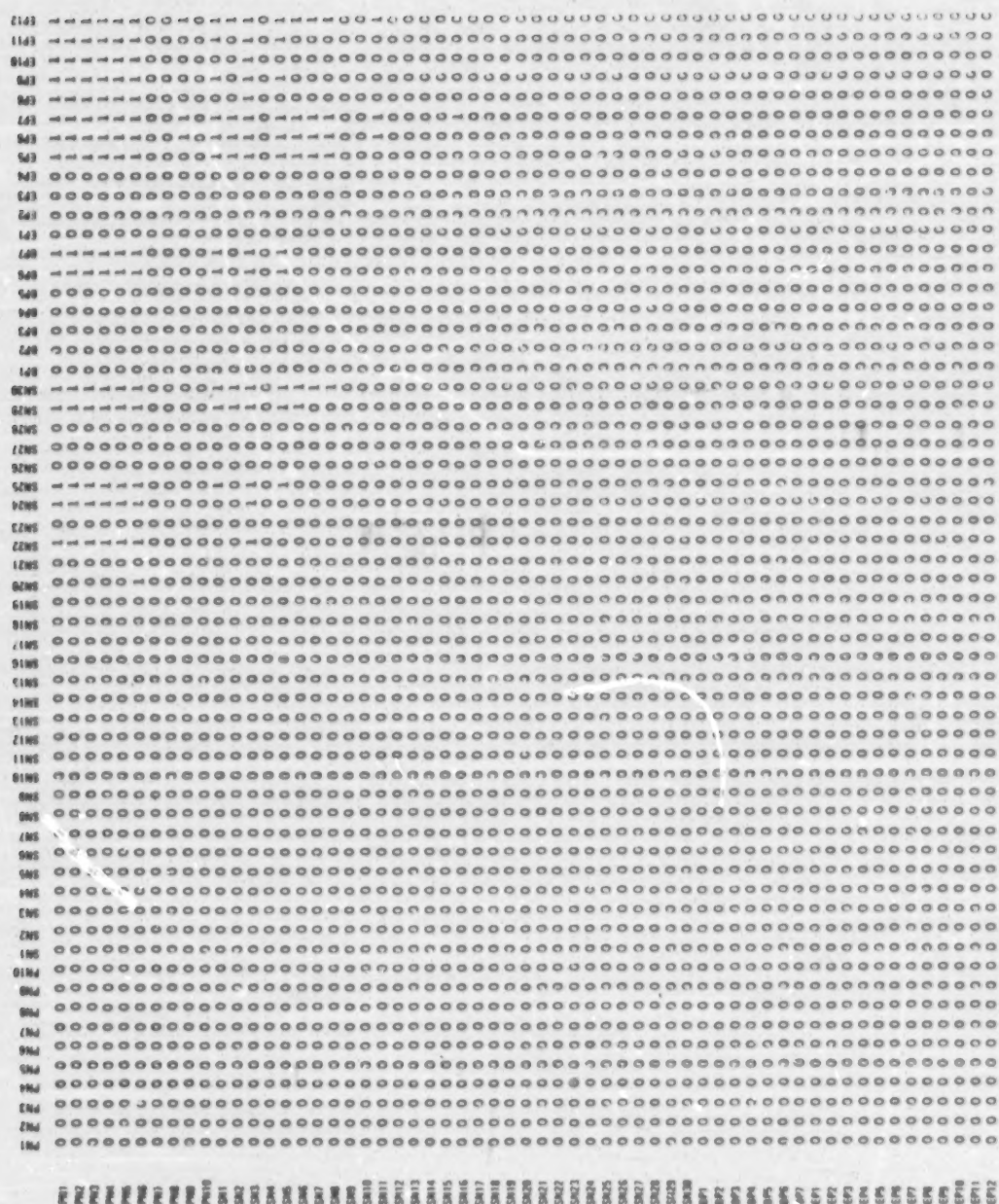


Figure 41. A⁹(D).

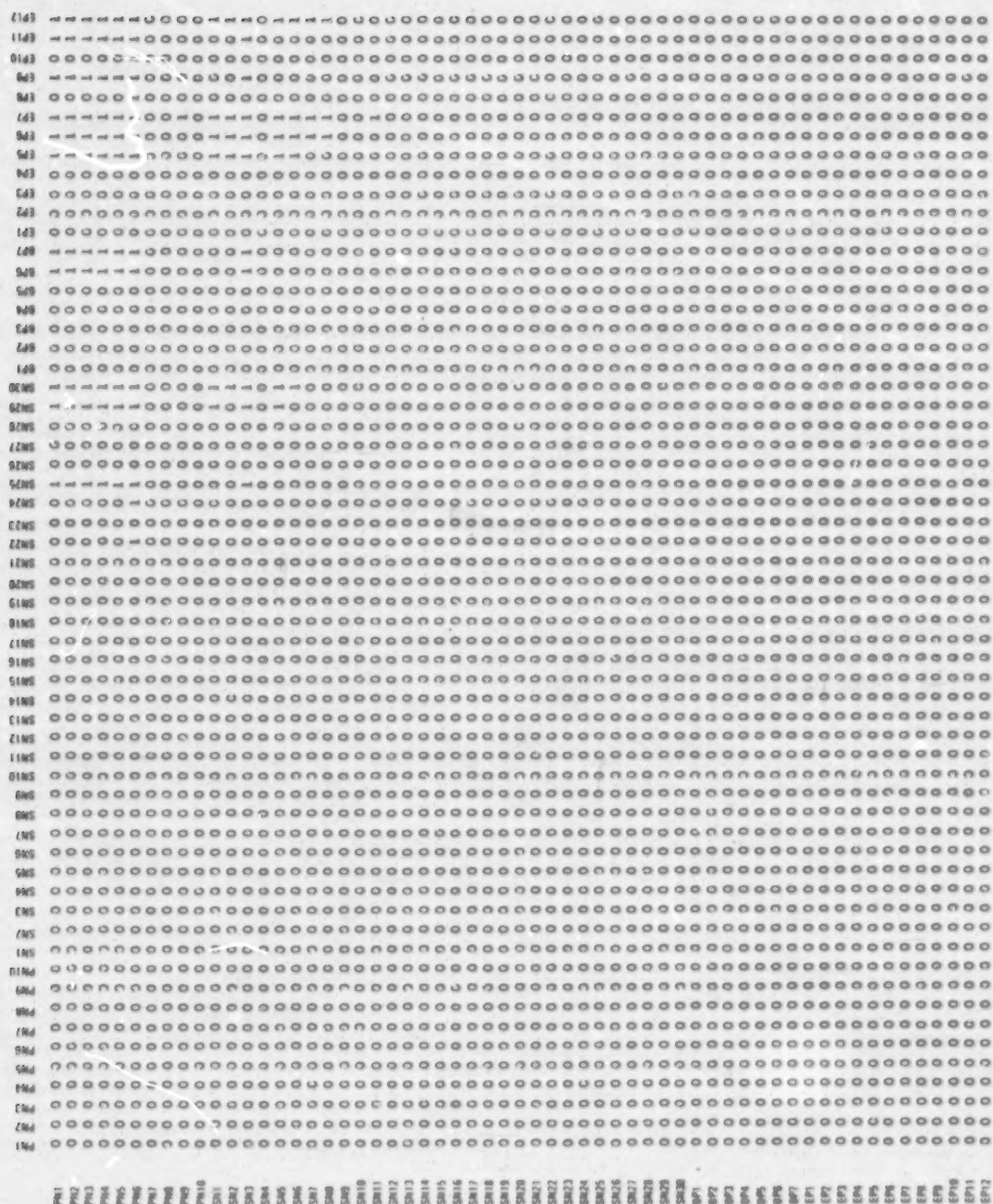


Figure 42. A^{10(D)}.

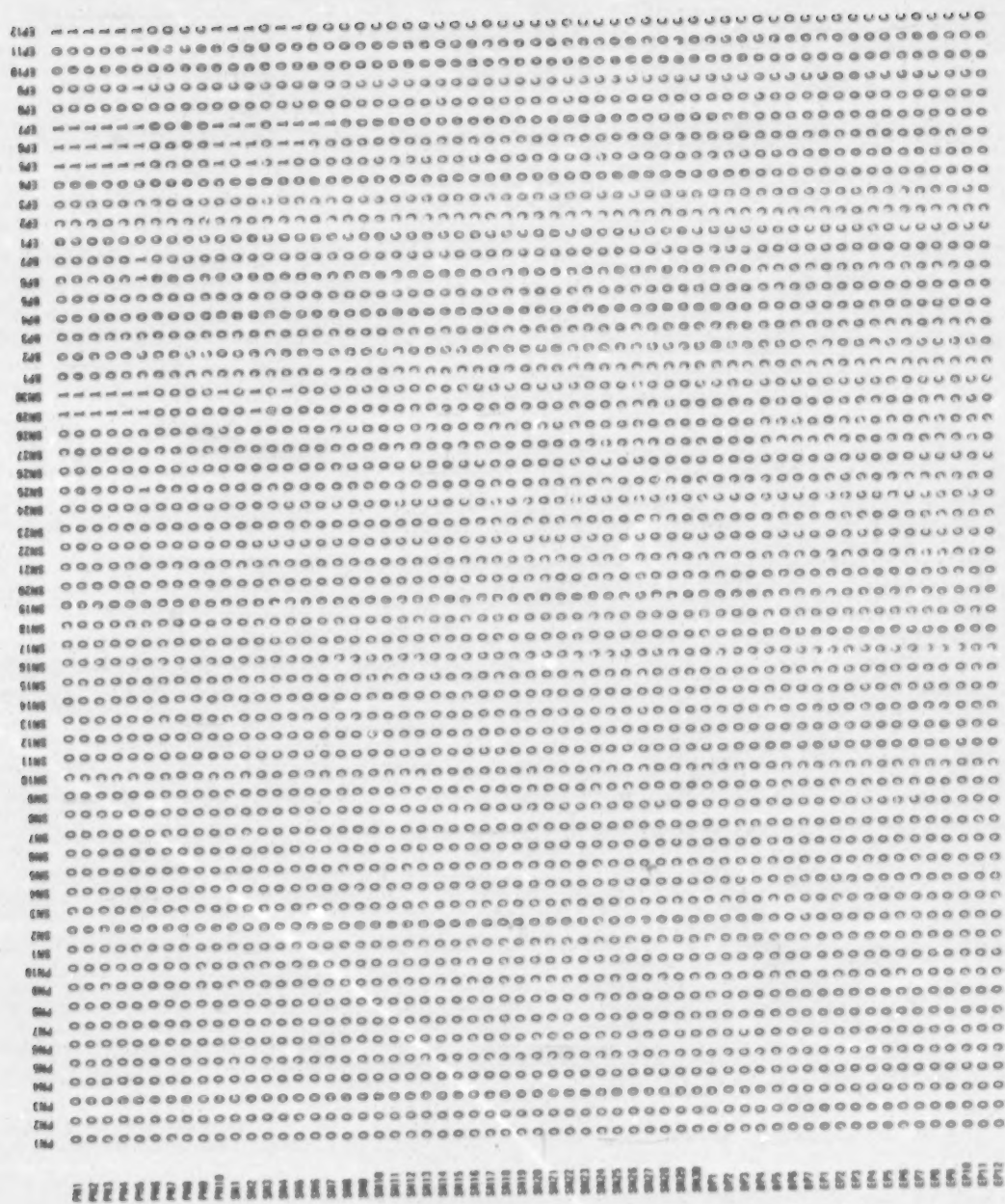
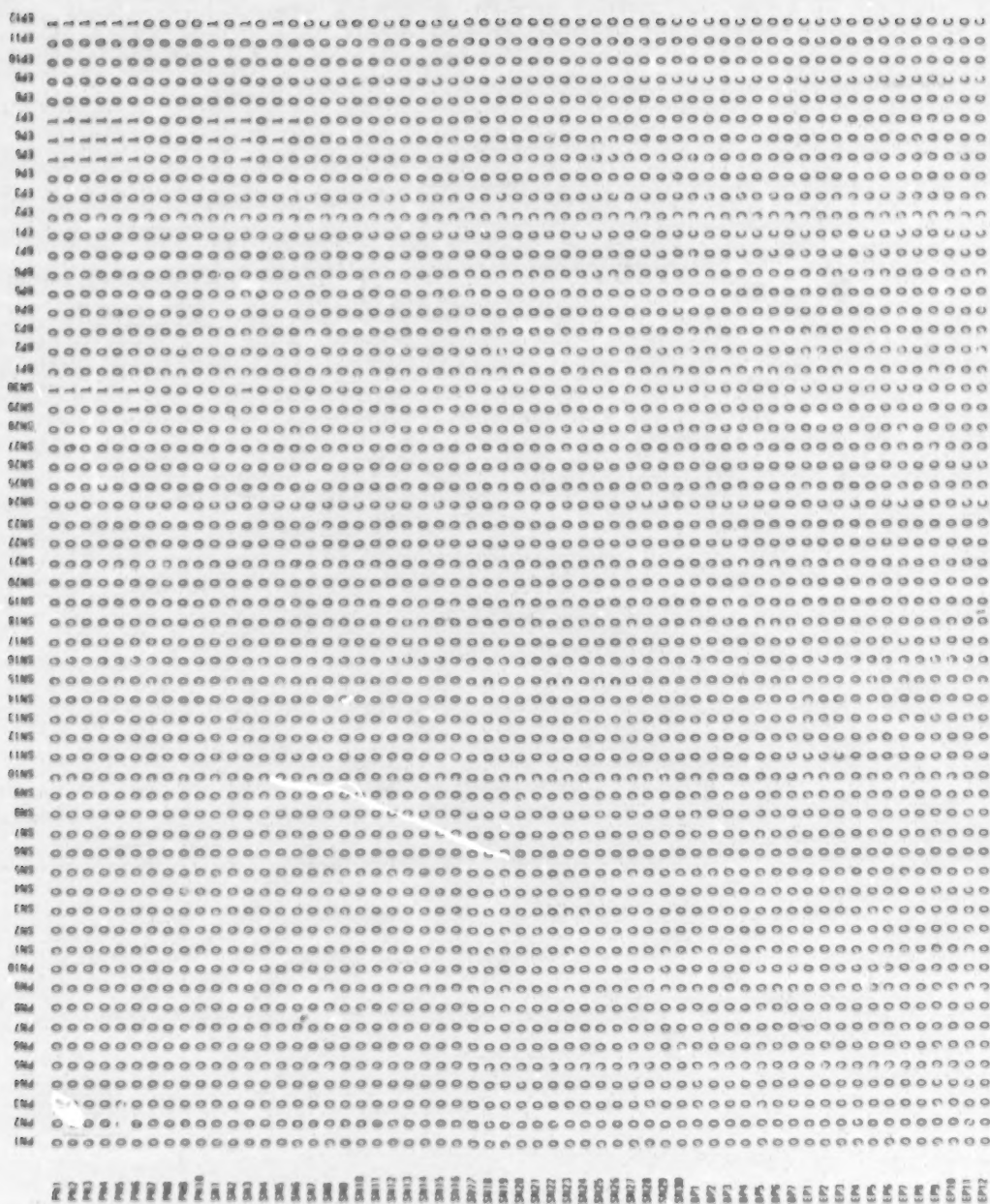


Figure 43. A¹¹(D).



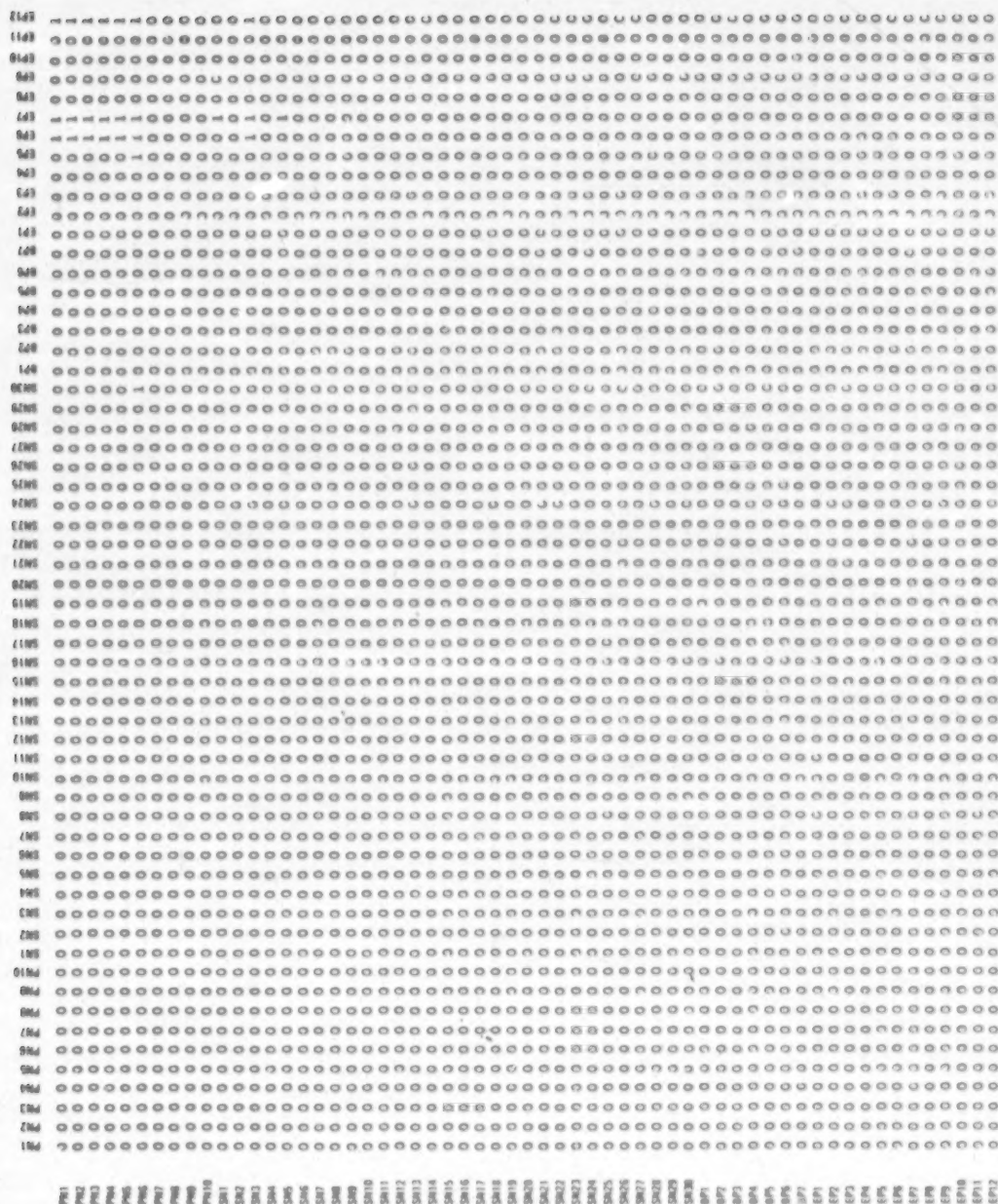
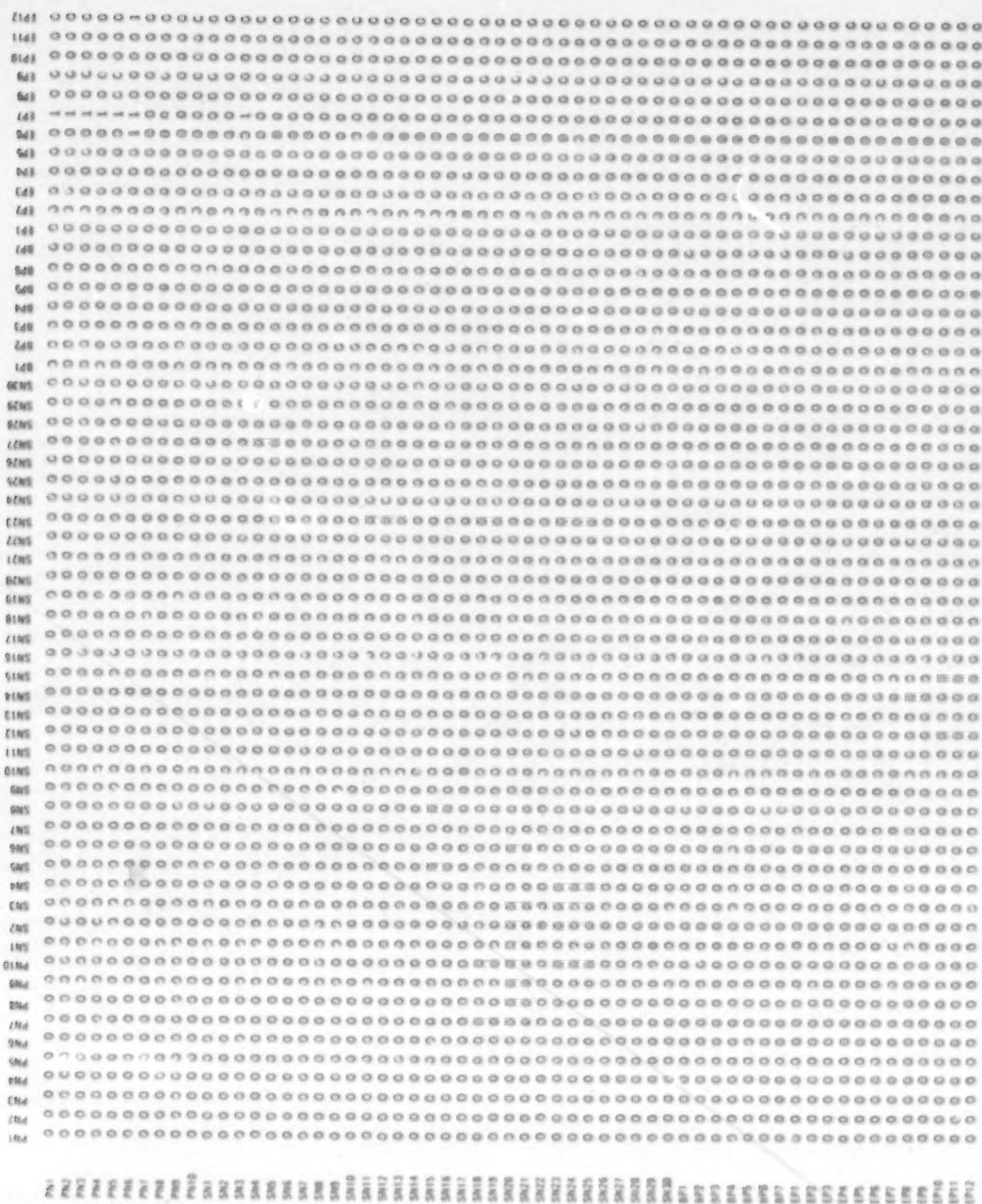


Figure 45. A¹³(D).



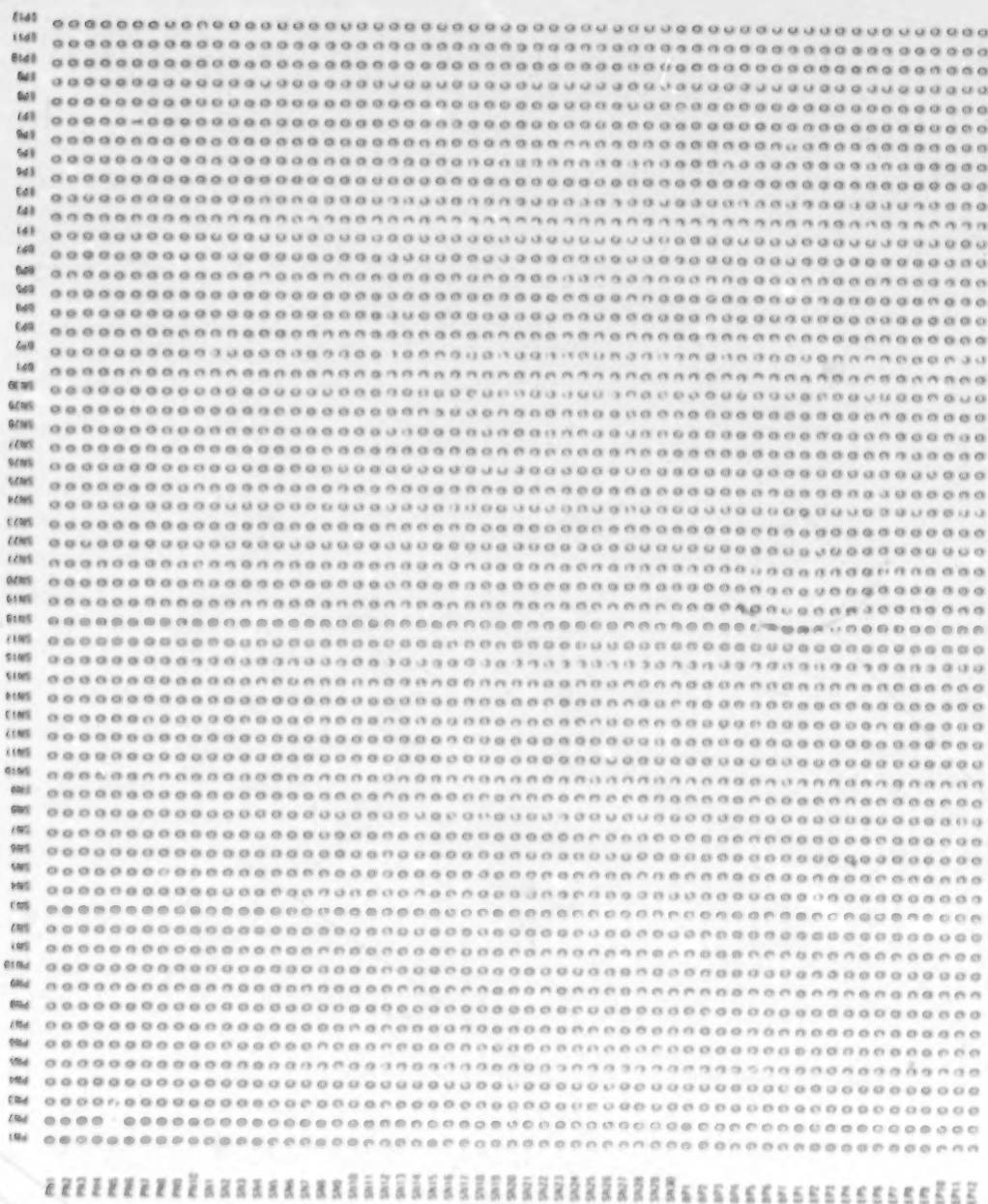
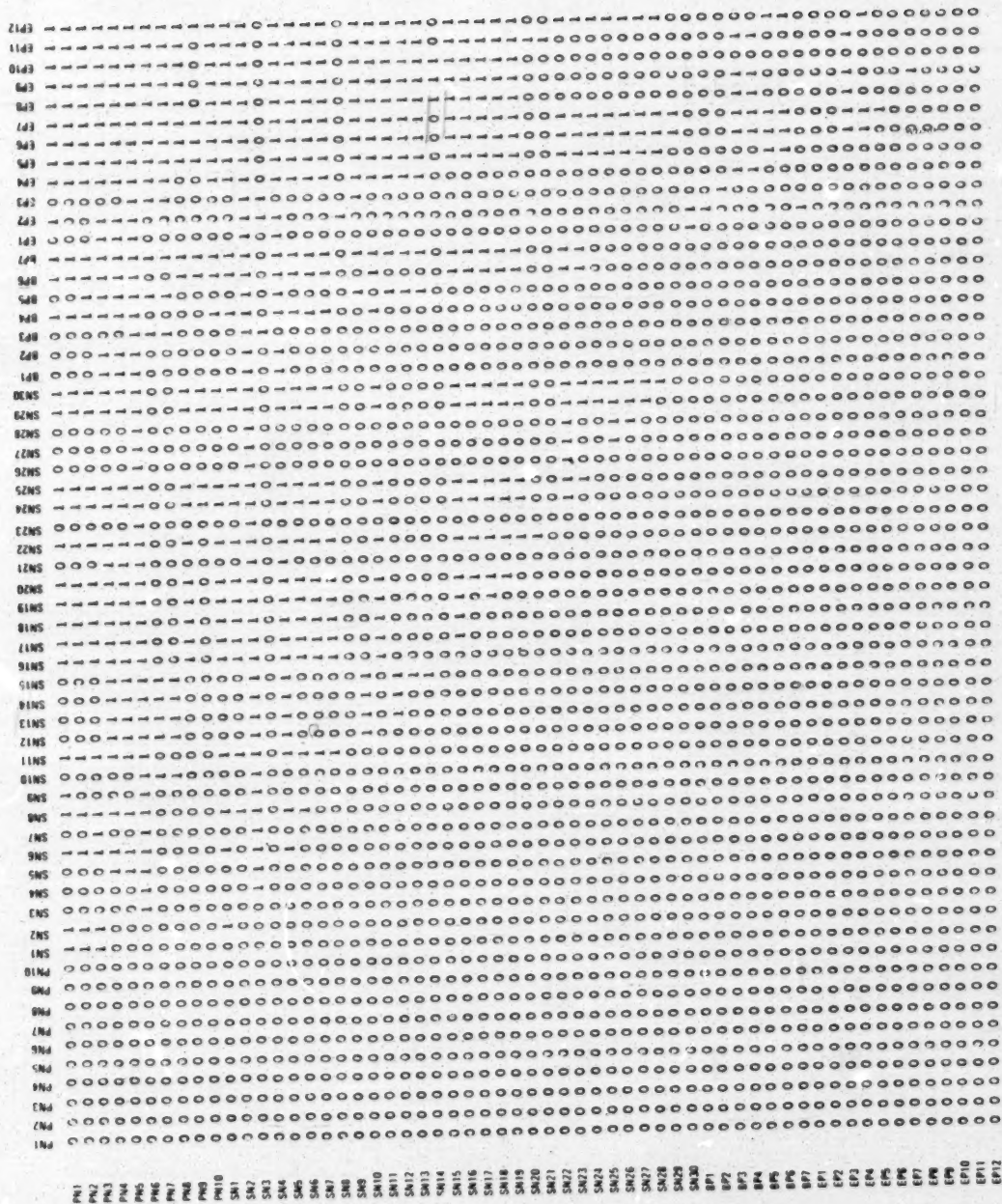


Figure 47. A 15/0.



TICS DEVELOPMENT

The next task is to develop the TICS for the photo course. The course material includes workbooks with each workbook having two parts. One part is a study guide consisting of questions that the student answers and corrects on his own as he progresses through the course. The second part is a multiple-choice test that is mailed to and corrected by the Extension Course Institute. As described in Chapter 1, the underlying idea for the TICS is that it is a collection of tasks used by the instructor to evaluate student comprehension of course material. Since the study guide is for the student and not the instructor's use, the study guide questions are not to be considered part of the TICS; thus, only the tests are part of the TICS.

In order to develop the TICS for the photo course, the only concern is with the multiple-choice test questions. The set of tasks T for the photo course consists of all the test questions. Every GCN constituent is associated with a class of sets of subsets of the set T . Recall that if c is a constituent, then F_c is the set of task sets associated with c . Recall also that if there are k task sets associated with c , then F_c is represented by

$$F_c = \{T_1^c, T_2^c, \dots, T_k^c\}$$

where the k elements T_i^c in F_c are task sets associated with c . In the photo course, the student only takes one test and takes the test once. This means there is only one criterion associated with each constituent: viz., the set of questions on that constituent. Hence, in the photo course for each GCN constituent c ,

$$F_c = \{T^c\}$$

where T^c is the set of questions associated with c .

The photo course TICS then is very simple. To develop the TICS, all one needs to do is to categorize the test questions as to which constituent each question is concerned with. For example, we find at least two questions associated with the Secondary Notion of a diaphragm.

- (1) The device that controls the intensity of the light that enters the camera is known as a (a) timer, (b) shutter, (c) lens diaphragm, or (d) bulb.
- (2) To close down a stop, you move the diaphragm from (a) f/11 to f/16, (b) f/16 to f/11, (c) f/1 to f/1.2, or (d) f/1.2 to f/1.

It should be clear from this brief discussion that the TICS for the photo course can be developed in a straightforward manner. Details of this effort are not necessary, since it is so obvious how to proceed.

STUDENT STATE DESCRIPTIONS

It remains to discuss the formulation of the student-state descriptions. In the photo course, there is no branching, or individualization, in the instruction. This means that no distinctions are made on the basis of nonsubject-matter attributes, such as IQ, reading level, and so forth. Thus, the only attributes to be considered are subject-matter attributes. The partial course analysis has identified 59 GCN constituents. Thus, there are 59 subject-matter-related attributes. This leads to 59 predicates.

P_{1x} : if and only if student x comprehends Primary Notion 1.

.

.

P_{10x} : if and only if student x comprehends Primary Notion 10.

P_{11x} : if and only if student x comprehends Secondary Notion 1.

.

.

P_{40x} : if and only if student x comprehends Secondary Notion 30.

P_{41x} : if and only if student x comprehends Basic Principle 1.

.

.

P_{47x} : if and only if student x comprehends Basic Principle 7.

P_{48x} : if and only if student x comprehends Established Principle 1.

.

.

P_{59x} : if and only if student x comprehends Established Principle 12.

Thus, we have 2^{59} possible student states. These states range from the state of comprehending no constituents

$$S_qx: \neg P_{1x} \ \& \ \neg P_{2x} \ \& \ \dots \ \& \ \neg P_{59x}$$

where $q = 2^{59}$, to the state of comprehending all constituents

$$S_{1x}: P_{1x} \ \& \ P_{2x} \ \& \ \dots \ \& \ P_{59x}$$

The goal of instruction is simply $G = \{S_1\}$.

Numerically, 2^{59} is a very large number of states, and it should be clear that a student will pass through only a small number of these states. In fact, ignoring degenerate loops consisting of remaining in the same state, the successful student passes through 59 states. This follows since the student learns one constituent at a time without ever going from comprehension to noncomprehension on a constituent. Thus, the student progresses in such a way that any path through the states consists of at most 59 states.

Consider the question of how many possible paths there are through the set of student states. As shown in Chapter 2, the GCN restricts the state transitions in two ways. First, some states are eliminated since they are impossible. These states are those that assert comprehension of one constituent, while asserting noncomprehension of one or more constituents on which the former depends. Second, the dependency relation can then be used to identify the possible paths through the set of possible student states.

In considering the partial GCN that has been identified for the photo course, it appears that there are a large number of possible student states. Unfortunately, at this juncture, exactly how many cannot be determined. The problem is that we have not been able to find a theoretical mechanism that can be used to identify possible states from among a large number of total states. In our case, it is simply not possible to go through the entire 2^{59} states eliminating the impossible states on the basis of the necessary and sufficient conditions given in Theorem 9 for a state to be a possible state.

We feel confident that such a mechanism can be formulated. However, at present, this development must be left as an important research problem that deserves attention. Unfortunately, this does not permit the identification of the total number of possible paths the student can take to master the photo course. Thus, we must admit that we have not been able to accomplish all the goals we set out to accomplish during this study. Additional work should lead to a solution of this problem.

Chapter 5

RELATION TO SOME OTHER THEORIES OF INSTRUCTION

The last component in Pask's list requires a representation of the teaching system, including teaching strategies. It is not possible to discuss this component in an axiomatic manner as was done with the first four Pask components. An axiomatization of a particular teaching system and its associated strategies could be devised but, at the present time, no immediately obvious way exists to defend the choice of one teaching system over another for axiomatization. Thus, instead of continuing the axiomatic theory, a less ambitious task is selected. The task is to show how the theory, developed in the preceding chapters, relates to some current theories of instruction. The theory of instruction selected for examining these relations is the one proposed by Merrill (18). Theories of instruction offered by Gagné (19) and Bloom (20) are not considered separately because Tennyson and Merrill (21) have shown these theories to be special cases of Merrill's theory.

THE MERRILL THEORY

Merrill defines 10 categories of behavioral outcomes in terms of the behavior implied and the psychological conditions under which the behavior may be observed. The ten categories that he thinks account for all learned behavior can be displayed by the diagram shown in Figure 49. The four levels of behavior, with the exception of the first level, are partitioned into three subdivisions by a measure of the degree of complexity. The first column, in the degree of complexity, is the simplest behavior within each level, and is characterized by a single response element to a particular stimulus situation. The second degree of complexity for each level represents a series of response elements to a given stimulus situation. The first two columns are thus characterized by a 1-1 situation. A many-1 situation characterizes the third degree of complexity. In this case, a student must respond differently to some stimulus set not previously encountered.

Merrill asserts the four levels to be hierarchical in the sense that behavior at a higher level includes some prerequisite behavior from each of the lower levels. For example,

Levels of Behavior	Emotional	Emotional (Signal Learning)		
	Psychomotor	Topographic (Stimulus Response)	Chaining (Chaining)	Complex Skill
	Memorization	Naming	Serial Memory (Verbal Association)	Discrete Memory (Multiple Discrimination)
	Complex Cognitive	Classification (Concept Learning)	Analysis (Principle Learning)	Problem Solving (Problem Solving)
		Single	Series	Set
		Degree of Complexity		

Figure 49. Ten categories of learned behavior (from Merrill, 18)

success in problem solving is a result of having gone through the prerequisite steps and obtaining the necessary skills at each level of behavior. In short, Merrill has proposed a hierarchical sequencing of the acquisition of various kinds of learned behavior. As Seidel¹ puts it, "...the hierarchy of instruction should virtually isomorphically reflect the hierarchy which may be appropriate for organizing the acquisition of various learning activities and learned behaviors."

RELATION OF SUBJECT-MATTER STRUCTURE THEORY TO MERRILL'S THEORY

The theory presented in the previous chapters can be related to Merrill's taxonomy of learning by assuming instruction to be isomorphic to the taxonomy. Table 16 is used to discuss the relationship. The top two rows of the table present a rearrangement of the information given in Figure 49 with the emotional category omitted. Three levels of behavior are the major column headings, and their subdivisions represent the three degrees of complexity. In each of the three columns, the degree of complexity follows the order: Single, Series, and Set. Rather than use these three terms, the category labels of Figure 49 are used instead.

¹Seidel, R. "A Meaningful Categorization of Computer Support for Instruction," Internal Memorandum. Instructional Technology Group, HumRRO Division No. 1, January 30, 1973.

The left-hand side of Table 16 lists the four notions of the subject-matter-structure theory discussed in Chapter 1. Check marks, in the cells of the table, show which categories of Merrill's taxonomy are involved for each of the four subject-matter notions. For example, if Primary Notions are being taught, topographic, naming, and classification categories are involved. If Secondary Notions are being taught, then six categories are involved in the following order: chaining, naming, serial memory, classification, and analysis. When Secondary Notions are being taught, it is assumed that the student has acquired the associated Primary Notions.

Table 16. Application of Subject-Matter Constituents to Merrill's Taxonomy

	Psychomotor			Memorization			Complex Cognitive		
	Topographic	Chaining	Complex Skill	Naming	Serial Memory	Discrete Memory	Classification	Analysis	Problem Solving
Primary Notions	✓			✓			✓		
Secondary Notions	✓	✓		✓	✓		✓	✓	
Basic Notions	✓	✓	✓	✓	✓	✓	✓	✓	
Established Principles	✓	✓	✓	✓	✓	✓	✓	✓	✓

In a later paper, Merrill (22, p. 109) presents a set of five assumptions on which he bases a task-analysis procedure. His paper has some interesting connections with the theoretical model presented in this report. The five assumptions Merrill gives are as follows:

- (1) Content and instructional strategies are independent phenomena.
- (2) Most courses particularly at the secondary or higher education level, involve only two types of content—concepts and operations.
- (3) Concepts and operations can be represented at two levels of abstraction—generalities and instances.
- (4) Most courses, particularly at the secondary or higher education level, involve only four levels of behavior—discriminated recall, classification, rule using, and rule finding.
- (5) Instructional strategies should revolve around rule-using or rule-finding tasks based on mastery models derived from needs and goals.

These five assumptions will be discussed, but not in the order they are listed. Starting with the second assumption, Merrill's "concepts" can be replaced by our

"constituents," and his "operations" can be replaced by our "relations." The two levels of abstraction, "generalities" and "instances," in Merrill's third assumption can be discussed in terms of the GCN and TICS. When the GCN constituents and relations are being presented by an instructor, they can be represented as generalities or instances. Similarly, the tasks in the associated TICS can be selected to represent generalities or instances. For certain subject matters, it may be possible to represent both generalities and instances in the GCN and in the TICS.

Merrill's fourth and fifth assumptions can be used to simplify the diagram shown in Table 16 by eliminating some of the categories. According to Merrill, the goals of most higher education courses primarily emphasize cognitive transfer behavior. Thus, instead of a large variety of behavior, Merrill finds that the goals are represented by four levels of behavior: discriminated recall, classification, rule-using, and rule-finding. These four levels of behavior can be integrated with the subject-matter structure theory of Chapter I and displayed by Table 17. The check marks in the cells indicate the level of behavior required for the GCN constituents shown on the left-hand side of the table.

Table 17. Application of Subject-Matter Constituents and Merrill's Four Levels of Behavior

	Discriminated Recall	Classification	Rule-Using	Rule-Finding
Primary Notions	✓			
Secondary Notions	✓	✓		
Basic Principles	✓	✓	✓	
Established Principles	✓	✓	✓	✓

We cannot agree with Merrill's first assumption. To elaborate on this disagreement, a starting point is furnished by a customary definition of instruction—instruction is the process of deliberately manipulating the environment of an individual so that learning can occur. In this definition, it appears reasonable to include the subject matter the student is to learn as part of his environment. In other words, both the content of a subject matter and its structure can be manipulated to permit learning, and, therefore, should be part of the environment. As noted in several places in this report, a subject matter does not have a unique GCN. The GCN depends upon what the instructor selects as GCN relations and how he classifies the constituents into Primary Notions, Secondary Notions, Basic Principles, and Established Principles.

In Chapter 3, it was shown how the GCN structure generates possible student-state transition diagrams that can be used to identify the number of possible paths through the GCN. In short, a strong connection was shown to exist between subject-matter constituents, its structure, and the possible paths through the structure. But, selecting a path through the GCN is an example of an instructional strategy.

If the designations of certain subject-matter constituents are changed, the GCN will be altered. In turn, this alteration will affect the permissible paths through the GCN, hence, the instructional strategies. For example, suppose a constituent in one organization of a subject matter is labeled a Primary Notion. Further, suppose this constituent is changed and relabeled as a Basic Principle in another organization of the subject matter. In this case, the two associated GCNs will be different, and the set of paths through the GCNs will be different.

In summary, the paths through a GCN are examples of instructional strategies and they are sensitive to changes in the GCN.

Chapter 6

RAMIFICATIONS

In the preceding chapters, an axiomatically based theory of subject-matter structure and a method of describing the state of a student were developed. In addition, it was shown how these two components can be combined to provide a state transition theory of instruction. At this juncture, a discussion will be given of some of the ramifications of this work. To simplify the discussion, the words "instructional agent" will be used to mean the intelligence that causes action to be taken in the instructional situation (Seidel and Kopstein, 23). The instructional agent operates in those areas of the instructional situation over which it has direct control. The instructional agent could thus be the course writer or the instructor (human or computer, or a combination).

An important implication of the material presented in the preceding chapters is that if an instructional agent applies the theory, then a great deal of clarity concerning the intended instruction must result. This is particularly true of any attempts to structure subject matter by using the axiomatic approach developed in Chapter 1. For example, to start applying the subject-matter structure theory, the instructional agent must begin to think in terms of the Primary Notions in the subject matter under consideration. To do this, forces the instructional agent to begin thinking very clearly about the underlying nature of the subject matter. Subsequently, identification of Secondary Notions, Basic Principles, and Established Principles reinforces and extends the clarity that is gained. Isolating and identifying subject-matter structuring relations (GCN relations), and the developing and understanding of the (GCN) dependency relation also result in a more clear understanding of the exact nature of the subject matter. This clarity concerning the subject matter is an extremely valuable asset with respect to providing coherent, effective instruction.

The process of constructing a GCN helps the instructional agent distinguish between "need-to-know" and "nice-to-know" constituents. This distinction can be made clear by examining the digraph representation of the GCN. For example, a constituent may be isolated, that is, a constituent with no arcs leading into it or leaving it. Such a constituent has to be re-examined. It could be a nice-to-know constituent and could be eliminated. To keep such an isolated constituent in the GCN requires that it be

connected to other constituents or combined with another constituent. Obvious benefits result when a GCN is used in conventional instruction. It helps to keep the instructor from wandering and digressing. The GCN can be given to a student. If given prior to instruction, it furnishes the student with a preview of the subject matter; if given after instruction, the GCN serves as a summary, and aids the student in reviewing.

The theory developed in this report is especially important as regards Computer Administered Instruction (CAI). In fact, it may be most applicable in a CAI environment. Complex subject matters (as noted in the Photo Course discussion), will result in extremely complex GCNs and very highly structured dependency relations. As a result, the number of possible student states, and the number of possible state transitions and paths may be very large. In fact, the instructional options concerning sequencing, and so forth, may be entirely too large for a noncomputerized instructional agent to grasp or even comprehend. In itself, this expanded realization can be considered as a definite contribution to the theory being discussed.

A CAI environment may be one of the few environments in which the implications of this instructional approach can be realized. Consider the GCN dependency relation as an example. The dependency relation can be easily stored in a computer as an array of 0s and 1s. A student-state description can also be easily stored in a computer for each of a number of students. This could be done as a one-dimensional array (vector) of 0s and 1s with each vector position corresponding to a particular attribute. With this information (the two arrays), a computer program can simply ascertain which options are available concerning sequencing for any student at any point in time. Note that this can be done simultaneously for a number of students.

In other words, a complete, comprehensive awareness concerning instructional options is available at any point in time for any individual student. If the theory developed in this report is applied, an instructional agent will probably come to realize that there are considerably more instructional options available than imagined. In a system-controlled environment, the system can be "aware" of more options, and can select the preferred option by some set of pre-programmed rules. In a learner-controlled environment, it would be possible to provide the learner with a more extensive "menu" of choices to select from. Thus, the options available concerning sequencing are identified more completely and can be more easily and fruitfully taken into consideration.

As an aid to developing effective instruction, these results are very important also. Suppose a complete understanding of possible student paths through some subject matter has been developed. By collecting data relating to paths traversed and subsequent

performance, some paths might be determined as less desirable than others. In this case, it might be decided to "close off" some of these paths. The path closure process would not be based upon subject-matter considerations, but upon other heuristic results. Such a result would only be forthcoming if there were a comprehensive awareness of potential instructional paths.

From a consideration of the dependency matrix, it may also be possible to gain an appreciation of those constituents crucial to a given subject matter. For example, the dependency matrix may show that two or three constituents are depended upon significantly more than others. In such a case, these constituents are in some sense pivotal, with respect to gaining comprehension of the subject matter. The instructional agent may decide in such cases to emphasize the understanding of these constituents. This may lead to special instruction, more detailed instruction, and so forth, concerning those constituents in order to more fully ensure student mastery of the subject matter.

It appears that the question of identifying pivotal constituents in a subject matter is an important problem in itself. It remains to be seen whether or not the theory developed can be extended to include such results. The mathematical approach, using matrices and graph theory, that has been followed does seem to provide a framework within which other theoretical results can be achieved. This fact must be considered to be an important result. A precise foundation for a theory of instruction has been formulated. It now seems possible to use this theory as a base from which other pressing instructional problems can be solved. Pask's fifth component, or the problem of representing instructional strategies, is one such problem that might be solved by extending the theory.

Chapter 7

OPEN PROBLEMS

Four classes of open problems were identified during the development of the theory and its application to the photo course:

- Problems encountered when the theory is applied to a subject matter containing many constituents.
- Theoretical problems.
- Instructional/learning implications.
- Comparisons with conventional methods of instruction.

Many of the problems in the four classes are interrelated. The motivation for obtaining solutions to these problems is to convert the theoretical model of this report into practical and useful instructional procedures.

Two large categories of open problems should be distinguished before discussing the above four classes. First, determinations have to be made of the difficulties in applying the theoretical model to a subject matter with many constituents—in other words, a subject matter representative of a full course. The first three classes of problems in the above list fall into this category. Second, a determination has to be made of the effects on the instructional/learning process. Stated somewhat differently, if the application of the theoretical model makes a difference in the instructional/learning situation, is the difference worth the time and effort involved? Problems in the fourth class in the above list fall in this category.

APPLICATION TO A FULL SUBJECT MATTER

A number of problems were encountered when the photo course was structured using the subject matter structure theory of Chapter 1. These problems are indicative of the types of problems to be expected when the theory is used to structure an entire subject matter.

The process used to structure the photo course started with isolating and listing the concepts in the text materials. Next, each of these concepts was categorized as a Primary

Notion, Secondary Notion, Basic Principle, or Established Principle. This initial categorization was revised a number of times when the relations, "used in the formulation of" and "used in the establishment of," were applied to connect the constituents. The result of this process was Tables 14 and 15, or equivalently, Figures 29 and 30.

The initial categorization of subject matter cannot be totally mechanized. It requires the efforts of subject matter experts. Some aid can be furnished by providing a set of written procedures to follow when structuring subject matter—that is, a manual explaining the axiomatic method and practical advice on how to apply the method to structuring subject matter.

A computer can be used in a variety of ways to help in the structuring process once initial adjacency matrices are constructed. Thus, a computer could be used to revise the matrices when constituents are added, deleted, or relabeled. Computer graphics could be used to display the digraph representation of the GCN. Programs could be developed to allow the course writer to see the entire digraph or selected portions of the digraph. Effects of altering the adjacency matrices on the digraphs could be seen immediately.

An open problem, related to the latter comments in the preceding paragraph, concerns the graphic representation of the GCN. Two representations are given in this report, the conventional digraph and the circuit diagram representation. It is probably necessary to develop other methods of graphical representation when the number of constituents in a subject matter is large. One possibility to explore is the graph-theoretic notion of the condensation of a graph (Harary, *et al.*, 13). To be most useful, methods of representing large graphs should be capable of computer graphic representation.

A final group of open application problems cannot be predicted. These are the problems not yet encountered, but bound to occur when a full-sized subject matter is to be structured.

THEORETICAL PROBLEMS

A main result of Chapter 3 showed how the subject matter structure is constrained by the dependency relation in two ways. First, some states are eliminated because they are impossible. Second, given a possible student state, potential successor states are limited by the dependency relation. Necessary and sufficient conditions were given for a

state to be a possible state. A procedure was also given that identified potential successor states and that led to a determination of all transition paths.

The results cited above have limited utility at present. Any reasonable subject matter can be expected to have numerous GCN constituents, perhaps several hundred. Assuming 200 constituents leads to 2^{200} total states, of which only a limited number are possible. Present derived results require an examination of each of these states to eliminate impossible states. A more effective procedure is needed to sort out impossible states. The process used to identify potential paths in the transition diagram is also cumbersome and, in many cases, perhaps not usable because of the large number of states. A more effective, limited procedure is needed to identify the transition paths.

The theoretical model presented in this report is more deterministic than probabilistic. Certain portions of the model are deterministic by their very nature. For example, GCNs produced by different individuals will most likely not be identical. However, any particular GCN is deterministic. The TIC associated with a particular GCN is not completely deterministic, because the number and degree of difficulty of the tasks can be easily changed.

In the present model, a student either comprehends or does not comprehend a constituent. At first glance, this formulation appears to be strictly deterministic. However, because of the flexibility in choosing tasks and task difficulty, the formulation is less than strictly deterministic or static. An open problem is the reformulation of a student's comprehension of constituents in terms of probability distributions. Such distributions would be conditional on the dependency chains involved and, most likely, also time dependent.

INSTRUCTIONAL IMPLICATIONS

The instructional implications of the results in this report were not explored in any detail. Some of the results have obvious bearings on the instruction/learning situation. However, it is an open problem to determine these bearings. The experimental aspects of this determination are considered in the next section. Some of the notions needing further investigation prior to experimentation are discussed in the following paragraphs.

Two graph-theoretic theorems were given to show how the adjacency matrix of the dependency relation could be used to compute the path length from one GCN constituent to another. Instructionally, such information appears to be important. For

example, being able to identify constituents with long dependency chains is of value. The dependency relation also shows some evidence of being useful as a diagnostic tool.

To illustrate, suppose a student fails to comprehend a constituent *c*. A diagnostic strategy would be to first identify the nearest constituents *c* depends upon—in other words, to identify the set of constituents connected to *c* by paths of length 1. If the student comprehends the constituents in this set, then the set of constituents nearest to them are identified. This is the set of constituents connected to *c* by paths of length 2. The process is continued until the constituents the student has forgotten are identified. The open problem for using the dependency relation as a diagnostic tool is the formulation of efficient computer-aided procedures for the gradual reconstruction of those parts of the GCN related to a particular constituent.

Another problem, discussed earlier, is concerned with an efficient procedure for determining the possible paths through a GCN. The instructional problem is to investigate the effects of presenting to students the different sequences of subject matter constituents—that is, to determine the effects on students by each of the possible paths through the GCN.

A final problem is to study the effects of different GCNs on student performance. In more detail, a subject matter does not have a unique GCN. Several subject matter experts, working independently, would probably produce different structures for the same subject matter. The open problem is to first determine how different GCNs can be constructed from different psychological bases. Given several such GCNs, the problem is then to determine their effects on the instruction/learning situation.

EXPERIMENTAL ASPECTS

Many of the problems discussed in the preceding sections can be resolved only by experimentation. Depending upon the problem, experiments would require all or some of the components: subject matters with many constituents, students, and instructional agents. Experiments using all three components could be done using course material presented conventionally, by correspondence, or with the aid of computers.

The first experimental efforts need to be directed toward determining and resolving the difficulties associated with structuring subject matter according to the theory presented in Chapter 1. After this problem is resolved, further experimentation should be undertaken to assess the effects on students of subject matter thus structured. In addition to determining this main effect, other assessments should be made of the instructional usefulness of other theoretical constructs presented in this report.

REFERENCES

1. Pask, G. "Computer-Assisted Learning and Teaching," paper presented at University of Leeds, England, 1969 (in press).
2. Kingsley, E. and Stelzer, J. "An Axiomatic Theory of Subject Matter Structure," Technical Report in preparation, Human Resources Research Organization.
3. Gagné, R.M. "The Acquisition of Knowledge," *Psychological Review*, vol. 69, 1962, pp. 355-365.
4. Gagné, R.M., Mayor, J.R., Garstens, H.L., and Paradise, N.E. "Factors in Acquiring Knowledge of a Mathematical Task," *Psychological Monographs*, no. 526, 1962.
5. Gagné, R.M. and Basaler, O.C. "Study of Retention of Some Topics of Elementary Nonmetric Geometry," *Educational Psychology*, vol. 54, 1963, pp. 123-131.
6. Tennyson, R.D. and Boutwell, R.C. "A Quality Control Design for Validating Hierarchical Sequencing of Programmed Instruction," *NSPI Journal*, vol. 10, May 1971, pp. 5-10.
7. Tuckman, B.W. and Edwards, K.J. "A Systems Model for Instructional Design and Management," *Educational Technology*, September 1971.
8. Hohn, F.E., Seshn, S., and Aufenkamp, D.D. "The Theory of Nets," *IRE Transactions on Electronic Computers*, September 1957, pp. 154-161.
9. Ashby, W.R. *An Introduction to Cybernetics*, John Wiley and Sons, New York, 1958.
10. Suppes, P. *Introduction to Logic*, Van Nostrand Reinhold Company, Princeton, 1957.
11. Gagné, R.M. and Paradise, N.E. "Abilities and Learning Sets in Knowledge Acquisition," *Psychological Monographs*, no. 518, 1961.
12. Gleason, A.M. *Fundamentals of Abstract Analysis*, Addison-Wesley Publishing Company, Reading, 1966.
13. Harary, F., Norman, R., and Cartwright, R. *Structural Models*, John Wiley and Sons, New York, 1965.
14. Hammer, P. and Rudeanu, S. *Boolean Methods in Operations Research*, Springer-Verlag, New York, 1968.
15. Seidel, Robert J. *Theories and Strategies Related to Measurement in Individualized Instruction*, Professional Paper 2-71, Human Resources Research Organization, March 1971.
16. Carnap, R. *The Continuum of Inductive Methods*, University of Chicago Press, Chicago, 1962.
17. Carnap, R. *The Logical Foundation of Probability*, (2nd ed.) University of Chicago Press, Chicago, 1963.
18. Merrill, M. David. "Necessary Psychological Conditions for Defining Instructional Outcomes," *Educational Technology*, August 1971, pp. 34-39.

19. Gagné, R.M. *The Conditions for Learning*, (1st ed.), Holt, Rinehart, and Winston, New York, 1965.
20. Bloom, B.S. (ed.) *Taxonomy of Educational Objectives, The Classification of Educational Goals, Handbook I: The Cognitive Domain*, David McKay Company, New York, 1956.
21. Tennyson, R.D. and Merrill, M.D. "Hierarchical Models in the Development of a Theory of Instruction: A Comparison of Bloom, Gagné and Merrill," *Educational Technology*, September 1971, pp. 27-31.
22. Merrill, M.D. "Content and Instructional Analysis for Cognitive Transfer Tasks," *Audiovisual Communications Review*, vol. 21, no. 1, Spring 1973, pp. 109-125.
23. Seidel, Robert J. and Kopstein, Felix F. *A General Systems Approach to the Development and Maintenance of Optimal Learning Conditions*, Professional Paper 1-68, Human Resources Research Organization, January 1968.

APPENDIX A—THE AXIOMATIC METHOD

The method we propose to use for developing a model of instruction parallels closely the axiomatic method of mathematics. The axiomatic method has a long and distinguished history beginning at least with Euclid some 2,000 years ago. As a discipline, in itself, axiomatics has only begun to be appreciated by mathematicians during this century. The pioneering efforts of mathematicians Russell and Whitehead in England and the logician Padoa in Italy, at the turn of the century, began to focus the attention of mathematicians on axiomatics as a separate discipline. At this point it will be appropriate to discuss briefly the axiomatic process as it is used in scientific endeavors.

Assume we have in mind a theory associated with a specific discipline. An axiomatization of the theory begins with the selection of a small number of properties from the theory that are judged to be basic in the theory. The selected properties are termed variously: primitive terms, undefined terms, primitive notions, or primitive concepts. The primitive terms are taken to be undefined, the only requirement is for them to be recognizable and distinguishable by their appearance. No attempts are made to analyze them further than this. They are taken to be basic intuitive notions on which the theory is built. The primitive terms of the axiomatization are usually described, or their intended meaning or interpretation is given, in general terms. The primitive terms are usually assumed to be independent of each other.

Considerable freedom often exists in the choice of the primitive notions for formalizing a theory. In other words, the intuitive content of a theory does not determine uniquely the primitive notions used in the axiomatization of the theory. In fact, it is usually possible to provide more than one axiomatization of the same theory. Each axiomatization may rest on a completely different set of primitive notions.

To illustrate, Euclidean plane geometry has been axiomatized in many different ways. One axiomatization uses the primitive notions: point, line, and a point lying on a line. These primitive notions are taken to be the intuitive, undefined bases for Euclidean geometry.

Other terms, concepts, and so forth, of a general discipline are introduced into the axiomatization of the discipline by definitions. An explicit definition relates the term to be defined (the definiendum) to other terms already available (the definiens). The definiens can consist solely of primitive terms, terms already defined, or a mixture of the two. However, the first new term defined in the axiomatization of the theory is given a meaning by relating it to some or all the primitive terms. The second new term can be defined similarly or by relating it to some of the primitive terms and the first definition. New terms defined in the axiomatization of the theory are thus individually introduced and generate a fixed sequence of definitions. It is, therefore, sensible to speak of the definitions preceding a particular definition.

It is usually required of any new term introduced in the theory by definition, that the new term be replaceable ultimately by a combination of the primitive terms in the

theory. In a sense, this requirement asserts that each defined term is a shortened notation for a longer expression. To construct an axiom system without this assumption would lessen the power that definitions have for facilitating deductive reasoning. Put another way, and using G.A. Miller's (A-1) term, definitions perform "chunking". This follows because a definition has at least two properties of chunking; it is a many-to-one process, and it is retrievable. In other words, a definition replaces a combination of many terms by a single term, and a defined term can always be decomposed into primitive terms. Questions of what logical conditions definitions must satisfy (the Theory of Definitions) are discussed in detail by Suppes (10).

An example of a Euclidean concept defined solely in terms of the set of primitive concepts (given above) is that of intersecting lines. Two lines intersect if there is a point lying on both lines. An example of a definition in terms of primitive notions and defined concepts is that of a triangle, namely, a triangle is a closed figure formed by three intersecting lines.

Primitive statements, also called axioms, are usually constructed from the set of primitive notions. They can also be formed by using defined terms or by a combination of primitive and defined terms. While this approach is not usual, it often makes the axioms simpler because of using defined terms. Axioms are thought of as statements fundamental to the theory being axiomatized, and they form the basis for deducing other statements of the theory. In general, there is a choice to be made between various sets of axioms with the normal requirement being that the axioms for a theory must be independent.

Axioms can be viewed as determining the meaning of the primitive terms of the theory by stating relations between the terms. Thus, the meaning, or sense, of the primitive terms is determined by the use made of them in the axioms. Such a view of specifying the meaning of a term is not an explicit definition, for it does not establish a relation between the new term (the definiendum) and older terms (the definiens). It can be regarded as an implicit definition because it acts like a definition by delimiting the meaning of a term.

Derived statements (also called theorems, deductions, propositions, and implications) are those statements that follow as a logical consequence of the primitive terms, defined terms, axioms, and previously derived statements. To be more precise, a statement is derived in the axiomatized version of the theory by a formal proof. A formal proof is a finite sequence of statements of the axiomatized theory such that each statement is either an axiom or is deducible from one or more preceding statements by logical rules of inference. Thus, a theorem is the last statement of some proof. An axiom can be viewed as a theorem with a one-step proof.

All axiomatizations of a theory are given within a body of presupposed knowledge. This aspect of axiomatization, although discussed last, is used throughout the axiomatization procedure. Thus, it is important to identify what other theories are assumed as known, even when selecting the primitive notions of the theory. Ordinarily, general set theory and some system of logic are presupposed as known in contemporary axiomatic formulations in mathematics. Standard portions of mathematics are assumed as known in addition to set theory and logic when axiomatizing an empirical science. The resulting axiomatization is called informal when standard logic and set theory are presupposed. The axiomatization of a theory is called formal when the rules of deduction and the system of logical axioms are not presupposed, but are given in a completely and

explicitly formalized language. Further discussion of the axiomatic method can be found in Suppes (10), Wilder (A-2, A-3), and Blanché (A-4).

Our general model is based upon the general principles involved in the axiomatic method. The main force of the axiomatic method is that a great deal of clarity is required in order to axiomatize a theory. The primitive notions must be clearly identified and well understood. The axioms must be sufficiently rich to capture the intended meaning of the theory and to allow for the derivation of all the expected results. Definitions and theorems must be clearly formulated.

Some characteristics of axiomatic systems are so restrictive that, if incorporated, they might hinder effective instruction. For example, in axiomatics, it is usually required that the primitive notions be independent and the axioms be independent. In fact, one of the main goals of axiomatics is to exhibit the most parsimonious structure in order to eliminate redundancy completely. From an instructional point of view, parsimony can be counterproductive since repetition and redundancy are often necessary and desirable. Our model does not require analogous restrictions that tend to create parsimony.

In summary, our axiomatic theory will possess a set of undefined notions, and all other notions of the theory will be defined in terms of these undefined ones. In addition, our theory will possess a set of axioms, and all remaining statements of the theory will be derived as consequences from these axioms.

REFERENCES FOR APPENDIX A

- A-1 Miller, G.A. *The Psychology of Communication*, Basic Books, Inc., New York, 1967.
- A-2 Wilder, R.L. *Introduction to the Foundations of Mathematics*, John Wiley and Sons, Inc., New York, 1967.
- A-3 Wilder, R.L. "The Role of the Axiomatic Method," *American Mathematical Monthly*, vol. 74, 1967, pp. 115-127.
- A-4 Blanché, R. *Axiomatics*, Dover Publications, New York, 1962.

APPENDIX B—PROOFS OF THEOREMS

Theorem 1. If x, y, z are GCN constituents and D the dependency relation, then

- (i) $\neg Dxx$
- (ii) $Dxy \ \& \ Dyz$ imply Dxz .

Proof:

(i) We show that the irreflexivity of D follows from the asymmetry of D . Symbolically, Axiom 6 reads Dxy implies $\neg Dyx$.

In this statement, replace y by x to get Dxx implies $\neg Dxx$. Assuming Dxx , thus leads to $\neg Dxx$, and the theorem is proved by *reductio ad absurdum*.

(ii) Suppose y depends on x and z depends on y . That is, suppose Dxy and Dyz . Then, by Definition 2, there exists GCN relations $R_1, R_2, \dots, R_n, S_1, S_2, \dots, S_m$ and GCN constituents $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m$, such that

$$(R_1xx_1 \ \& \ R_2x_1x_2 \ \& \ \dots \ \& \ R_nx_ny) \ \& \ (S_1yy_1 \ \& \ S_2y_1y_2 \ \& \ S_my_mz).$$

By Definition 2, this expression asserts that Dxz , or that z depends on x .

Theorem 2. If x and y are GCN constituents, and R a GCN relation, then

- (i) $\neg Rxx$
- (ii) Rxy implies $\neg Ryx$.

Proof:

(i) Suppose that Rxx holds. Then, by Definition 2, it follows that x depends upon itself, and this is contrary to part (i) of Theorem 1.

(ii) Suppose Ryx is true, then by the transitivity of the dependency relation, $Rxy \ \& \ Ryx$ imply Rxx , and this contradicts part (i) of the theorem.

Theorem 3. If $x_1, x_2, \dots, x_n$ are GCN constituents, and $R_1, R_2, \dots, R_n$ are GCN relations, then it is impossible to have a sequence of the form

$$R_1x_1x_2 \ \& \ R_2x_2x_3 \ \& \ \dots \ \& \ R_nx_nx_1.$$

Proof: The existence of the above sequence, by Definition 2, leads to the conclusion that x_1 depends on x_1 , and this contradicts part (i) of Theorem 1.

Theorem 4.

- (i) Let x be a Primary Notion in a GCN. Then $\neg Ryx$ for every constituent y and relation R in the GCN.
- (ii) Let y be a Secondary Notion in a GCN. Then Rxy holds for some constituent x , where x is a Primary Notion or a Secondary Notion, and R is some relation in the GCN.
- (iii) Let x be a Basic Principle in a GCN. Then $\neg Ryx$ for every constituent y that is a Basic Principle or Established Principle in the GCN and every GCN relation R .
- (iv) Let y be an Established Principle in a GCN. Then Rxy holds for some constituent x and some relation R in the GCN.

Proof:

(i) Suppose Ryx , then by Definition 2, this implies Dyx , which contradicts Axiom 1.

(ii) Suppose $\neg Rxy$, then by Definition 2, this implies $\neg Dxy$. This last statement asserts that a Secondary Notion does not depend upon a Primary or Secondary Notion, and contradicts Axiom 2.

(iii) Suppose Ryx , then Dyx by Definition 2. But, Dyx asserts that a Basic Principle depends upon a Basic Principle or Established Principle, and is contrary to Axiom 3.

(iv) Suppose $\neg Rxy$, then $\neg Dxy$, by Definition 2. But, $\neg Dxy$ asserts that an Established Principle does not depend upon any other notion contrary to Axiom 4.

Theorem 6. Let x, y be GCN constituents, then y is dependent on x if and only if $a^*_{xy} = 1$.

Proof: By definition, y is dependent on x if and only if there is a sequence of GCN relations leading from x to y . Such a sequence will be composed of n steps or links. Thus, the entry representing the x, y relation in the n^{th} power of the total adjacency matrix $A^*(D)$ will be 1. When the dependency matrix is formed by adding all non-zero adjacency matrices, using Boolean arithmetic, this entry will carry over causing a^*_{xy} to be set to 1.

Theorem 8. If an individual comprehends a GCN constituent y , then the individual must comprehend all GCN constituents on which y depends.

Proof: Let y be a constituent that an individual comprehends, and let x be any constituent on which y depends. By Definition 2, there exists a chain of GCN constituents and relations.

$$R_{1xx_1} \ \& \ R_{2x_1x_2} \ \& \ \dots \ \& \ R_{n+1x_ny}.$$

Because the individual comprehends y , there is a task set $T_i^y \in F_y$, such that the individual can perform all the tasks in T_i^y . By Axiom 9, there exists a task set $T_k^{\alpha_n} \in F_{x_n}$, such that the individual can perform all tasks in $T_k^{\alpha_n}$. Continuing in this way, after $n+1$ applications of Axiom 9, we have a set of tasks T_j^x , such that $T_j^x \in F_x$, and the individual can perform all tasks in T_j^x . By definition, the individual comprehends x , as required.

Theorem 9. For all $i = 1, \dots, 2^n$, S_i is a possible student state if and only if constituent c_j depends on constituent c_k , then $P_j \in \text{APRED}(S_i)$ implies $P_k \in \text{APRED}(S_i)$, where P_j is associated with c_j , P_k is associated with c_k , and $j, k = 1, \dots, m$.

Proof: (Necessity) Suppose S_i is a possible student state. Assume also that c_j depends on c_k , $P_j \in \text{APRED}(S_i)$, but $P_k \notin \text{APRED}(S_i)$. This means P_k is negated in S_i . Thus, by definition, S_i is an impossible student state, which is contrary to our initial assumption.

(Sufficiency) Suppose for all c_j, c_k ($j, k = 1, \dots, m$), if c_j depends on c_k , then

$$P_j \in \text{APRED}(S_i) \text{ implies } P_k \in \text{APRED}(S_i).$$

Assume also that S_i is an impossible student state. Since S_i is impossible, we know there is some c_j, c_k such that c_j depends on c_k , P_j is unnegated in S_i , and P_k is negated in S_i . This means that

$$P_j \in \text{APRED}(S_i)$$

and

$$P_k \notin \text{APRED}(S_i).$$

But, since P_j depends on P_k , we can show by assumption that

$$P_k \in \text{APRED}(S_i)$$

which is contradictory, thus establishing the possibility of state S_i .

Theorem 10: For all $i, j = 1, \dots, 2^n$ if $S_i, S_j \in \bar{S}$ then $S_i \Rightarrow S_j$ if and only if either

- (i) $\text{APRED}(S_i) = \text{APRED}(S_j)$ or
- (ii) $\text{APRED}(S_j) = \text{APRED}(S_i) \cup \{P_k\}$

where $P_k \notin \text{APRED}(S_i)$, $k = 1, \dots, m$.

Proof: (Necessity) Suppose $S_i \Rightarrow S_j$.

By definition, any student in state S_j comprehends *at least* the same constituents as a student in state S_i . This simply means that

$$* \text{APRED}(S_i) \subseteq \text{APRED}(S_j).$$

Furthermore, any student in state S_j comprehends *at most* one more constituent than the student comprehended in state S_i . Thus, there are two possibilities. One, the student comprehends no more constituents in S_j than in S_i , in which case (*) becomes $\text{APRED}(S_i) = \text{APRED}(S_j)$, which is (i) in the theorem. Or, on the other hand, comprehension of exactly one new constituent is gained. This means there is some c_k , such that P_k associated with c_k is such that $P_k \notin \text{APRED}(S_i)$, but $P_k \in \text{APRED}(S_j)$. Thus, $\text{APRED}(S_j) = \text{APRED}(S_i) \cup \{P_k\}$, which is (ii) in Theorem 2 as required.

(Sufficiency) Suppose first that

$$\text{APRED}(S_i) = \text{APRED}(S_j).$$

Any student in state S_i comprehends only those constituents associated with the elements in $\text{APRED}(S_i)$. Similarly, any student in state S_j comprehends only those constituents associated with the elements in $\text{APRED}(S_j)$. But, since $\text{APRED}(S_i) = \text{APRED}(S_j)$, it follows that any student in state S_i comprehends exactly the same constituents as any student in state S_j . Thus, $S_i \Rightarrow S_j$ as required.

Suppose next, that $\text{APRED}(S_j) = \text{APRED}(S_i) \cup \{P_k\}$, when $P_k \notin \text{APRED}(S_i)$ and $k = 1, \dots, m$. Consider any student in state S_j . This student comprehends the constituents associated with the elements in $\text{APRED}(S_j)$. Another student in state S_i comprehends the constituents associated with the elements in $\text{APRED}(S_i)$. Since it can be seen that

$$\text{APRED}(S_i) \subseteq \text{APRED}(S_j),$$

it follows that the student in state S_j comprehends at least the same constituents as the student in state S_i . But the student in state S_j also comprehends a constituent c_k associated with P_k where $P_k \notin \text{APRED}(S_i)$. Thus, the student in state S_j comprehends exactly one more constituent than the student in state S_i . It can be seen that this single constituent is the most that the comprehension of these two students can differ. Thus, $S_i \Rightarrow S_j$ as required.